Recall that when all outcomes of an experiment are equally likely, the probability of an event E is $P(E) = \dfrac{n(E)}{n(S)}$. The event E **and** F occurs when *both* E and F occur. In this handout we learn how to find the probability of such an event, especially when E and F describe two different stages of an experiment, such as two tosses or two draws.

Before we state the rule, let us see it in action.

Example 1. A fair coin is tossed and a fair die is rolled. Find the probability of getting
a) heads and a 6, b) tails and an even number.

Solution. An outcome is a pair (coin, die). There are 2 results for the coin and 6 for the die, so there are $2 \cdot 6 = 12$ equally likely outcomes:

$$H1 \quad H2 \quad H3 \quad H4 \quad H5 \quad H6$$
$$T1 \quad T2 \quad T3 \quad T4 \quad T5 \quad T6$$

a) Only one outcome, $H6$, is heads and a 6. Notice that the product of the two probabilities gives the same answer:

$$P(\text{heads and } 6) = \frac{1}{12} \qquad \text{and} \qquad P(\text{heads}) \cdot P(6) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}.$$

b) Three outcomes, $T2$, $T4$, $T6$, are tails and an even number. Again,

$$P(\text{tails and even}) = \frac{3}{12} = \frac{1}{4} \qquad \text{and} \qquad P(\text{tails}) \cdot P(\text{even}) = \frac{1}{2} \cdot \frac{3}{6} = \frac{1}{4}.$$

In both parts, the probability of "E and F" is the product of the two probabilities. This works because the coin and the die do not influence each other.

Definition. Two events are **independent** if the occurrence of one does not change the probability of the other.

Separate tosses of a coin, separate rolls of a die, and drawing *with* replacement (putting the item back before the next draw) all lead to independent events.

The AND Rule for Independent Events. If E and F are independent, then

$$P(E \text{ and } F) = P(E) \cdot P(F).$$

The same is true for three or more independent events: we multiply all of their probabilities.

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Example 2. A fair die is tossed twice. Find the probability that

a) both tosses are 6, b) the first toss is even and the second toss is greater than 4.

Solution. The two tosses are independent.

a) $P(6 \text{ and } 6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

b) $P(\text{even, then greater than } 4) = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. We can check this by counting: $3 \cdot 2 = 6$ of the 36 outcomes have an even first toss and a 5 or 6 on the second toss, and $\frac{6}{36} = \frac{1}{6}$.

Example 3. A fair coin is tossed four times. Find the probability that all four tosses are heads.

Solution. The four tosses are independent, so

$$P(\text{four heads}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

Example 4. A basketball player makes 80% of her free throws. Assume that her shots are independent. She takes two free throws. Find the probability that she makes

a) both shots, b) neither shot, c) exactly one shot.

Solution. She makes a shot with probability 0.8 and misses with probability $1 - 0.8 = 0.2$.

a) $P(\text{both}) = 0.8 \cdot 0.8 = 0.64$ b) $P(\text{neither}) = 0.2 \cdot 0.2 = 0.04$

c) The three results "both," "neither," and "exactly one" are mutually exclusive, and one of them must happen. So their probabilities add up to 1, and

$$P(\text{exactly one}) = 1 - 0.64 - 0.04 = 0.32.$$

Dependent Events

When we draw *without* replacement, the first draw changes what is left for the second draw. Such events are not independent; we call them **dependent**.

Example 5. Two cards are drawn from a standard deck of 52 cards, one after the other, without replacement. Find the probability that both cards are aces.

Solution. The probability that the first card is an ace is $\frac{4}{52}$. If the first card is an ace, then 51 cards are left, and 3 of them are aces. So the probability that the second card is also an ace is $\frac{3}{51}$.

$$P(\text{both aces}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$$

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In this solution, the second fraction is the probability that the second card is an ace *given that* the first card was an ace. In general, we write $P(F \text{ given } E)$ for the probability that F occurs when we already know that E has occurred.

The AND Rule. For any two events E and F,

$$P(E \text{ and } F) = P(E) \cdot P(F \text{ given } E).$$

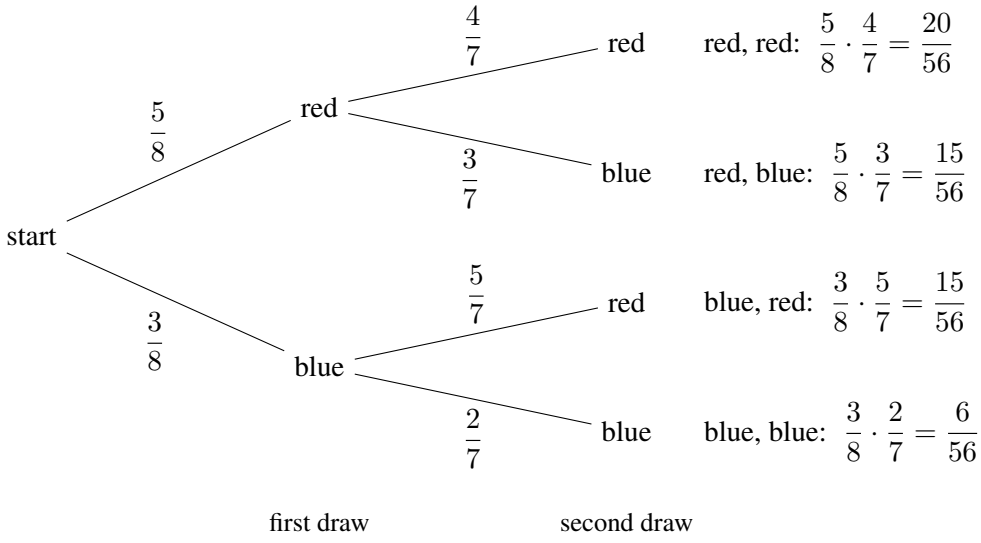
If E and F are independent, then $P(F \text{ given } E) = P(F)$, and we get back the AND rule for independent events.

A **tree diagram** is a good way to organize a two-stage experiment. We write the probabilities of the first stage on the first branches and the probabilities of the second stage, given what happened first, on the second branches. The probability of a path is the product of the probabilities along it.

Example 6. A bag contains 5 red and 3 blue marbles. Two marbles are drawn without replacement. Find the probability that

a) both marbles are red, b) the marbles have different colors.

Solution. The tree diagram shows every possible result. After a red marble is drawn, 4 red and 3 blue marbles are left. After a blue marble is drawn, 5 red and 2 blue marbles are left.



The four path probabilities add up to $\dfrac{20+15+15+6}{56} = 1$, as they should.

a) $P(\text{both red}) = \dfrac{5}{8} \cdot \dfrac{4}{7} = \dfrac{20}{56} = \dfrac{5}{14}$

b) Two paths give different colors: red then blue, or blue then red. These paths are mutually exclusive, so we add their probabilities:

$$P(\text{different colors}) = \frac{15}{56} + \frac{15}{56} = \frac{30}{56} = \frac{15}{28}.$$

Be careful. With replacement, the draws are independent, and the answer to part a) would be $\frac{5}{8} \cdot \frac{5}{8} = \frac{25}{64}$ instead. Always check whether the first draw changes the second one.

"At Least One" Problems

The event "at least one" has many cases, but its complement, "none," has only one. So we use the complement rule, $P(\text{not } E) = 1 - P(E)$, together with the AND rule.

Example 7. A fair die is rolled four times. Find the probability of getting at least one 6.

Solution. The complement of "at least one 6" is "no 6 on any of the four rolls." Each roll is not a 6 with probability $\frac{5}{6}$, and the rolls are independent.

$$P(\text{at least one 6}) = 1 - \left(\frac{5}{6}\right)^4 = 1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.518$$

This is a little better than an even chance. In the 17th century, the French gambler Chevalier de Méré won money betting on this event.

Example 8. (The birthday problem) How many people must be in a room so that the probability that at least two of them have the same birthday is at least 50%? Ignore February 29, and assume that all 365 birthdays are equally likely.

Solution. The complement of "at least two people share a birthday" is "all birthdays are different." Imagine the people entering the room one at a time. The first person can have any birthday. The second person must avoid one birthday, so the probability is $\frac{364}{365}$. The third person must avoid two birthdays, with probability $\frac{363}{365}$, and so on. For n people,

$$P(\text{at least two share a birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - n + 1}{365}.$$

With a calculator, we find these values:

number of people	10	20	22	23	30	40	50
probability of a shared birthday	0.117	0.411	0.476	0.507	0.706	0.891	0.970

So 23 people are enough: with 22 people the probability is still below 50%, but with 23 people it is about 50.7%. Most people guess a much larger number. The surprise is that we are not asking whether someone shares *your* birthday, but whether *any two* of the 23 people share a birthday, and there are $\frac{23 \cdot 22}{2} = 253$ pairs of people who could match (just like 253 handshakes among 23 people).

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Practice Problems

1. A fair coin is tossed and a fair die is rolled. Find the probability of getting
   a) tails and a number greater than 4, b) heads and an odd number.
2. A fair die is tossed twice. Find the probability that
   a) both tosses are even, b) the first toss is a 1 and the second is not, c) neither toss is a 6.
3. A fair coin is tossed five times. Find the probability of getting
   a) five tails, b) at least one head.
4. Two cards are drawn from a standard deck of 52 cards. Find the probability that
   a) both are hearts, if the cards are drawn without replacement,
   b) the first is a king and the second is a queen, without replacement,
   c) both are hearts, if the first card is put back before the second draw.
5. A bag contains 6 red and 4 green marbles. Two marbles are drawn without replacement.
   a) Draw a tree diagram for this experiment.
   b) Find the probability that both marbles are green.
   c) Find the probability that the first marble is red and the second is green.
   d) Find the probability that the marbles have different colors.
6. The probability of rain is 0.3 on Saturday and 0.4 on Sunday. Assume that the two days are independent. Find the probability that it rains
   a) on both days, b) on neither day, c) on at least one of the two days.
7. A quiz has 4 multiple-choice questions, each with 5 choices. A student guesses on every question. Find the probability that the student gets
   a) all four correct, b) none correct, c) at least one correct.
8. Decide whether the two events are independent. Explain.
   a) A card is drawn and put back, then a second card is drawn. E: the first card is red. F: the second card is red.
   b) Two cards are drawn without replacement. E: the first card is red. F: the second card is red.

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Answers

1. a) $\frac{1}{2} \cdot \frac{2}{6} = \frac{1}{6}$ b) $\frac{1}{2} \cdot \frac{3}{6} = \frac{1}{4}$

2. a) $\frac{3}{6} \cdot \frac{3}{6} = \frac{1}{4}$ b) $\frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$ c) $\frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36}$

3. a) $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$ b) $1 - \frac{1}{32} = \frac{31}{32}$

4. a) $\frac{13}{52} \cdot \frac{12}{51} = \frac{1}{17}$ b) $\frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652} = \frac{4}{663}$ c) $\frac{13}{52} \cdot \frac{13}{52} = \frac{1}{16}$

5. a) First draw: red $\frac{6}{10}$, green $\frac{4}{10}$. After red: red $\frac{5}{9}$, green $\frac{4}{9}$. After green: red $\frac{6}{9}$, green $\frac{3}{9}$.
   b) $\frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$ c) $\frac{6}{10} \cdot \frac{4}{9} = \frac{24}{90} = \frac{4}{15}$ d) $\frac{24}{90} + \frac{24}{90} = \frac{48}{90} = \frac{8}{15}$

6. a) $0.3 \cdot 0.4 = 0.12$ b) $0.7 \cdot 0.6 = 0.42$ c) $1 - 0.42 = 0.58$

7. a) $\left(\frac{1}{5}\right)^4 = \frac{1}{625}$ b) $\left(\frac{4}{5}\right)^4 = \frac{256}{625}$ c) $1 - \frac{256}{625} = \frac{369}{625}$

8. a) Independent. The first card is put back, so the deck is the same for the second draw: $P(F) = \frac{26}{52}$, no matter what the first card was.
   b) Dependent. The second card is red with probability $\frac{25}{51}$ if the first card is red, but $\frac{26}{51}$ if the first card is black.

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