Recall that when all outcomes of an experiment are equally likely, the probability of an event E is $P(E) = \dfrac{n(E)}{n(S)}$. In many problems we select a *group* of objects at random: a committee, a hand of cards, or a few items from a box. In these problems the order of selection does not matter, so both $n(E)$ and $n(S)$ are numbers of combinations.

Combinations. The number of ways to choose k objects out of n different objects, when the order does not matter, is

$$\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k!}.$$

For example, $\dbinom{12}{3} = \dfrac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220.$

Example 1. A study group has 12 students: 5 of them are math majors, and 7 are not. Three students are selected at random to present a problem. Find the probability that

a) all three are math majors, b) exactly two are math majors, c) none of them is a math major.

Solution. The sample space is the set of all possible groups of 3 students out of 12:

$$n(S) = \binom{12}{3} = 220$$

a) To select three math majors, we choose 3 of the 5 math majors.

$$P(\text{all three are math majors}) = \frac{\dbinom{5}{3}}{\dbinom{12}{3}} = \frac{10}{220} = \frac{1}{22} \approx 0.045$$

b) Now we make two decisions: 2 of the 5 math majors, *and* 1 of the 7 other students. By the multiplication principle, we multiply the two numbers.

$$P(\text{exactly two math majors}) = \frac{\dbinom{5}{2}\dbinom{7}{1}}{\dbinom{12}{3}} = \frac{10 \cdot 7}{220} = \frac{70}{220} = \frac{7}{22} \approx 0.318$$

c) We choose all 3 students from the 7 students who are not math majors.

$$P(\text{no math majors}) = \frac{\dbinom{7}{3}}{\dbinom{12}{3}} = \frac{35}{220} = \frac{7}{44} \approx 0.159$$

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Another way. We could also solve part a) with the AND rule, by selecting the students one at a time without replacement:

$$P(\text{all three are math majors}) = \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} = \frac{60}{1320} = \frac{1}{22}$$

We get the same answer. Both methods work, but the AND rule gets complicated when the selected objects are of different kinds, as in part b). Combinations handle such problems easily.

Example 2. A box contains 20 light bulbs, and 4 of them are defective. Three bulbs are selected at random. Find the probability that

a) none of them is defective, b) at least one of them is defective.

Solution. There are 4 defective and 16 good bulbs, and $\binom{20}{3} = 1140$ ways to select 3 bulbs.

a) All three bulbs must come from the 16 good ones.

$$P(\text{none defective}) = \frac{\binom{16}{3}}{\binom{20}{3}} = \frac{560}{1140} = \frac{28}{57} \approx 0.491$$

b) "At least one" has three cases (1, 2, or 3 defective bulbs), but its complement, "none," has only one. So we use the complement rule.

$$P(\text{at least one defective}) = 1 - \frac{28}{57} = \frac{29}{57} \approx 0.509$$

Example 3. A bag contains 5 red, 4 blue, and 3 green marbles. Three marbles are selected at random. Find the probability that the three marbles have three different colors.

Solution. There are 12 marbles, so $n(S) = \binom{12}{3} = 220$. We need one marble of each color: 1 of the 5 red, 1 of the 4 blue, and 1 of the 3 green marbles.

$$P(\text{three different colors}) = \frac{\binom{5}{1}\binom{4}{1}\binom{3}{1}}{\binom{12}{3}} = \frac{5 \cdot 4 \cdot 3}{220} = \frac{60}{220} = \frac{3}{11} \approx 0.273$$

Another way. This problem can also be solved with the AND rule, but we have to be careful: the three colors can come in any order. For example, the probability of red, then blue, then green is $\frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} = \frac{60}{1320} = \frac{1}{22}$. There are $3! = 6$ possible orders of the three colors, and each order has the same probability, $\frac{1}{22}$. So $P(\text{three different colors}) = 6 \cdot \frac{1}{22} = \frac{6}{22} = \frac{3}{11}$, the same answer.

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Cards and Lotteries

A standard deck has 52 cards: 13 in each of the four suits (clubs, diamonds, hearts, and spades), with 4 cards of each value (four aces, four kings, and so on). A five-card hand is a selection of 5 cards, so there are $\binom{52}{5} = 2\,598\,960$ possible hands.

Example 4. Five cards are dealt from a well-shuffled deck. Find the probability that

a) all five cards are hearts, b) the hand contains exactly two aces.

Solution. a) We choose all 5 cards from the 13 hearts.

$$P(\text{all hearts}) = \frac{\binom{13}{5}}{\binom{52}{5}} = \frac{1\,287}{2\,598\,960} \approx 0.000495$$

That is about 1 chance in 2000.

b) We choose 2 of the 4 aces, and 3 of the 48 cards that are not aces.

$$P(\text{exactly two aces}) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{6 \cdot 17\,296}{2\,598\,960} = \frac{103\,776}{2\,598\,960} \approx 0.040$$

Example 5. In a lottery, the player selects 6 different numbers from 1 to 49. Then 6 winning numbers are drawn at random. Find the probability that the player

a) wins the jackpot (all six numbers are correct), b) has exactly 3 correct numbers.

Solution. The number of ways to draw 6 numbers is $\binom{49}{6} = 13\,983\,816$.

a) Only one of these draws matches the player's numbers.

$$P(\text{jackpot}) = \frac{1}{13\,983\,816} \approx 0.0000000715$$

That is about 1 chance in 14 million.

b) Think of the 49 numbers as two groups: the player's 6 numbers and the other 43. Exactly 3 correct numbers means that 3 of the winning numbers come from the player's 6, and the other 3 come from the remaining 43.

$$P(\text{exactly 3 correct}) = \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} = \frac{20 \cdot 12\,341}{13\,983\,816} = \frac{246\,820}{13\,983\,816} \approx 0.018$$

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Splitting into Groups

Example 6. a) In how many ways can we split 6 people into two groups of three?

b) The 6 people are split into two groups of three at random. Find the probability that two given people, Anna and Ben, end up in the same group.

Solution. a) It is tempting to say $\binom{6}{3} = 20$: we choose 3 people for the first group, and the other 3 form the second group. But this counts every split twice. For example, if the six people are A, B, C, D, E, and F, then choosing $\{A, B, C\}$ and choosing $\{D, E, F\}$ both give the same split:

$$\{A, B, C\} \text{ and } \{D, E, F\}$$

So the number of splits is

$$\frac{\binom{6}{3}}{2} = \frac{20}{2} = 10.$$

We can check this in another way. Let us look at the group of person A. We need to choose 2 more people for this group from the other 5, and every choice gives a different split: $\binom{5}{2} = 10$.

b) Let us look at the group of Anna again. There are 10 possible splits. Anna and Ben are in the same group if Anna's group contains Ben and one more person, who can be any of the other 4 people. So 4 of the 10 splits are good.

$$P(\text{Anna and Ben are in the same group}) = \frac{4}{10} = \frac{2}{5} = 0.4$$

Be careful. If the two groups have names, for example a red team and a blue team, then $\{A, B, C\}$ on the red team and $\{A, B, C\}$ on the blue team are different splits. In that case, the number of splits is $\binom{6}{3} = 20$, and we do not divide by 2.

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Practice Problems

1. A committee of 4 people is selected at random from 6 teachers and 8 students. Find the probability that the committee consists of

   a) students only, b) exactly 2 teachers and 2 students, c) at least one teacher.

2. Five cards are dealt from a standard deck of 52 cards. Find the probability that

   a) all five cards are red, b) the hand contains exactly 3 kings, c) the hand contains no aces.

3. In a lottery, 5 different numbers are drawn from 1 to 40. A player selects 5 numbers. Find the probability that the player

   a) has all 5 numbers correct, b) has exactly 4 numbers correct.

4. A box contains 15 batteries, and 3 of them are dead. Four batteries are selected at random. Find the probability that

   a) none of them is dead, b) exactly one of them is dead.

5. A bag contains 6 red, 4 blue, and 2 green marbles. Four marbles are selected at random. Find the probability that

   a) there are 2 red and 2 blue marbles, b) all four marbles have the same color.

6. An exam has 10 questions, and 4 of them are about probability. A student selects 3 questions at random to answer. Find the probability that none of the selected questions is about probability.

7. Solve Problem 1a again, using the AND rule instead of combinations.

8. Eight people are split into two groups of four.

   a) In how many ways can the eight people be split into a red team and a blue team, with four people on each team?

   b) In how many ways can the eight people be split into two groups of four if the groups have no names?

   c) If the eight people are split into two groups of four at random, what is the probability that two given friends end up in the same group?

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Answers

1. a) $\dfrac{\binom{8}{4}}{\binom{14}{4}} = \dfrac{70}{1001} = \dfrac{10}{143} \approx 0.0699$ b) $\dfrac{\binom{6}{2}\binom{8}{2}}{\binom{14}{4}} = \dfrac{420}{1001} = \dfrac{60}{143} \approx 0.420$ c) $1 - \dfrac{10}{143} = \dfrac{133}{143} \approx 0.930$

2. a) $\dfrac{\binom{26}{5}}{\binom{52}{5}} = \dfrac{65780}{2598960} \approx 0.0253$ b) $\dfrac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} = \dfrac{4512}{2598960} \approx 0.00174$

   c) $\dfrac{\binom{48}{5}}{\binom{52}{5}} = \dfrac{1712304}{2598960} \approx 0.659$

3. a) $\dfrac{1}{\binom{40}{5}} = \dfrac{1}{658008} \approx 0.00000152$ b) $\dfrac{\binom{5}{4}\binom{35}{1}}{\binom{40}{5}} = \dfrac{175}{658008} \approx 0.000266$

4. a) $\dfrac{\binom{12}{4}}{\binom{15}{4}} = \dfrac{495}{1365} = \dfrac{33}{91} \approx 0.363$ b) $\dfrac{\binom{3}{1}\binom{12}{3}}{\binom{15}{4}} = \dfrac{660}{1365} = \dfrac{44}{91} \approx 0.484$

5. a) $\dfrac{\binom{6}{2}\binom{4}{2}}{\binom{12}{4}} = \dfrac{90}{495} = \dfrac{2}{11} \approx 0.182$ b) $\dfrac{\binom{6}{4} + \binom{4}{4}}{\binom{12}{4}} = \dfrac{16}{495} \approx 0.0323$

6. $\dfrac{\binom{6}{3}}{\binom{10}{3}} = \dfrac{20}{120} = \dfrac{1}{6} \approx 0.167$

7. $\dfrac{8}{14} \cdot \dfrac{7}{13} \cdot \dfrac{6}{12} \cdot \dfrac{5}{11} = \dfrac{1680}{24024} = \dfrac{10}{143}$, the same answer as in Problem 1a.

8. a) $\binom{8}{4} = 70$ b) $\dfrac{\binom{8}{4}}{2} = \dfrac{70}{2} = 35$ c) $\dfrac{\binom{6}{2}}{\binom{7}{3}} = \dfrac{15}{35} = \dfrac{3}{7} \approx 0.429$

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