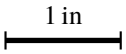
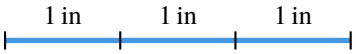
Perimeter is a measure of length. The length of an object, measured in inches, is the number of 1-inch line segments we need to cover our object.



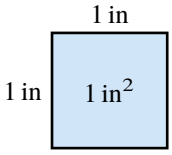
unit inch



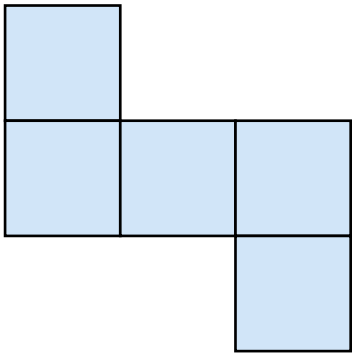
This blue line is 3 inches long.

Lengths are measured in meters (m), inches (in), feet (ft), miles (mi), and so on.

Area is a measure of surface. The area of an object, measured in square inches, is the number of 1-inch by 1-inch squares we need to cover our object.



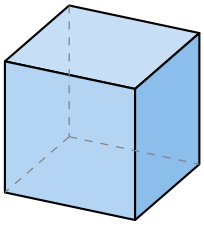
1 square inch



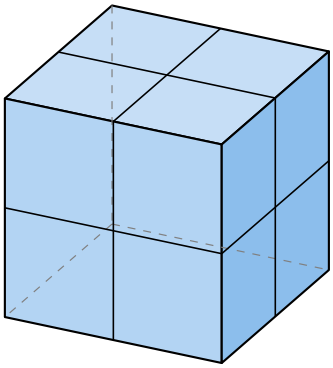
This area is 5 square inches.

Area is measured in square meters (m^2), square inches (in^2), square feet (ft^2), and so on.

Volume is a measure of space. The volume of an object, measured in cubic inches, is the number of 1-inch by 1-inch by 1-inch cubes we need to build a replica of our object. Another way to imagine this is to make an ice-sculpture replica of our object, melt it, and then freeze it into unit cubes and count them.



1 cubic inch

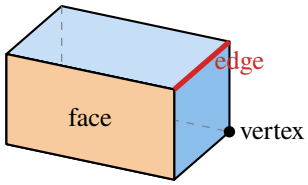


The volume of this object is 8 cubic inches.

Volumes are measured in cubic meters (m^3), cubic inches (in^3), cubic feet (ft^3), and so on.

Another way of thinking of all this is that length is one-dimensional (part of a line), area is two-dimensional (part of a plane), and volume is three-dimensional (part of space). The space we live in is three-dimensional.

Definition: The flat surfaces of a solid like a box are its **faces**. The line segments where two faces meet are its **edges**. The corner points, where edges meet, are its **vertices** (singular: **vertex**). For example, a box has 6 faces, 12 edges, and 8 vertices.



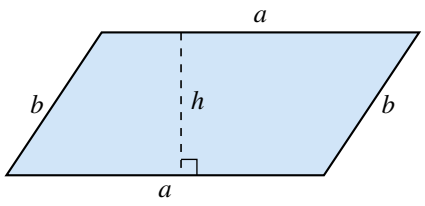
An Easy Way to Remember Volume Formulas

Let us call a shape a **solid** if it starts in some shape and ends in the same shape, and a **pointy** object if it starts in a shape and ends in a point. In the plane, for example, a parallelogram is a solid and a triangle is a pointy object. In space, a cylinder is a solid and a cone is a pointy object.

2-dimensional solid

Area is base times height

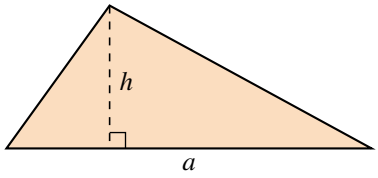
$$A = ah$$



2-dimensional pointy

Area is base times height over 2

$$A = \frac{ah}{2}$$

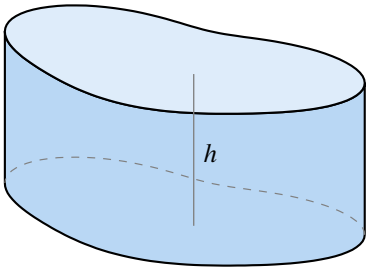


3-dimensional solid

Volume is base area times height

$$V = Bh$$

where B is the area of the base

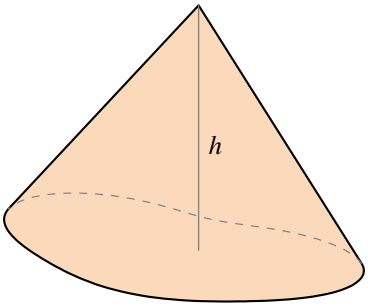


3-dimensional pointy

Volume is base area times height over 3

$$V = \frac{Bh}{3}$$

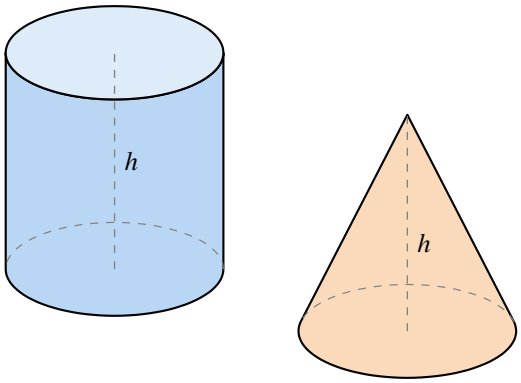
where B is the area of the base



Maybe we cannot visualize objects in four dimensions, but their "volume" formula is probably $\frac{Vh}{4}$: base volume times the height, over four. Either way, the point here is that using this general rule is easier than memorizing many volume formulas. Always remember that the base area B must be computed first, and that the height h is measured at a right angle to the base.

Example 1. The solid in the picture is called a **cylinder**. The pointy object is called a **cone**. Find the volume of each if the base is a circle with area $30\,\text{in}^2$ and the height is $12\,\text{in}$.

Solution: The area of the base is given, so we can apply our new volume formulas right away.

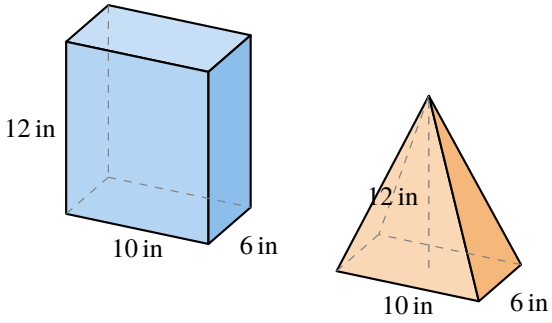


$$V_{\text{cylinder}} = Bh = 30\,\text{in}^2 \cdot 12\,\text{in} = \boxed{360\,\text{in}^3}$$

$$V_{\text{cone}} = \frac{Bh}{3} = \frac{30\,\text{in}^2 \cdot 12\,\text{in}}{3} = \boxed{120\,\text{in}^3}$$

Example 2. The solid in the picture is a **rectangular prism**. The pointy object is a **rectangular pyramid**. Find the volume of each.

Solution: In both cases, the base is a rectangle with sides $10\,\text{in}$ and $6\,\text{in}$, so its area is $B = 10\,\text{in} \cdot 6\,\text{in} = 60\,\text{in}^2$. The height is, in both cases, $12\,\text{in}$.



$$V_{\text{prism}} = Bh = 60\,\text{in}^2 \cdot 12\,\text{in} = \boxed{720\,\text{in}^3}$$

$$V_{\text{pyramid}} = \frac{Bh}{3} = \frac{60\,\text{in}^2 \cdot 12\,\text{in}}{3} = \boxed{240\,\text{in}^3}$$

Example 3. Find the volume of a cylinder and of a cone, if the base is a circle of radius $3\,\text{in}$ and the height is $10\,\text{in}$. Give the exact answer and an approximation rounded to the nearest thousandth.

Solution: The area of the base is $B = \pi r^2 = \pi\,(3\,\text{in})^2 = 9\pi\,\text{in}^2$.

$$V_{\text{cylinder}} = Bh = 9\pi\,\text{in}^2 \cdot 10\,\text{in} = \boxed{90\pi\,\text{in}^3 \approx 282.743\,\text{in}^3}$$

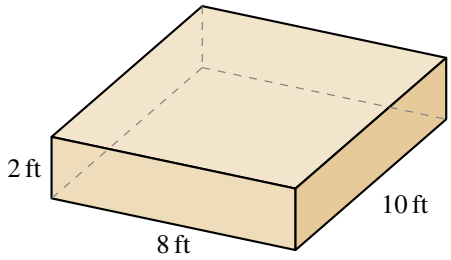
$$V_{\text{cone}} = \frac{Bh}{3} = \frac{9\pi\,\text{in}^2 \cdot 10\,\text{in}}{3} = \boxed{30\pi\,\text{in}^3 \approx 94.248\,\text{in}^3}$$

Example 4. A cylindrical water tank has a diameter of 6 feet and a height of 5 feet. How many cubic feet of water does it hold when it is full?

Solution: The diameter is 6 feet, so the radius is 3 feet. The area of the base is $B = \pi\,(3\,\text{ft})^2 = 9\pi\,\text{ft}^2$, and

$$V = Bh = 9\pi\,\text{ft}^2 \cdot 5\,\text{ft} = \boxed{45\pi\,\text{ft}^3 \approx 141.372\,\text{ft}^3}$$

Example 5. A sandbox is 8 feet long and 10 feet wide, and the sand in it is 2 feet deep. How many cubic feet of sand are in the sandbox? How many 2-cubic-foot bags of sand were needed to fill it?

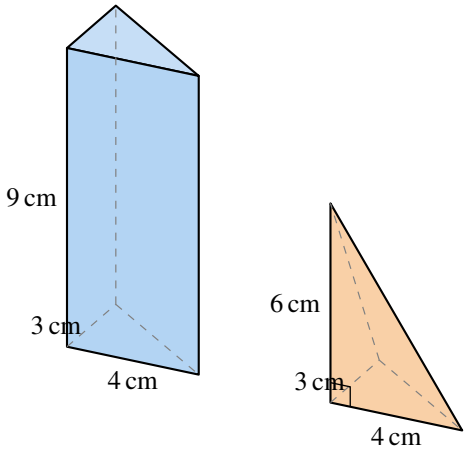


Solution: The sand forms a rectangular prism. Its base is a rectangle, 8 feet by 10 feet, so $B = 80\,\text{ft}^2$, and its height is the depth of the sand.

$$V = Bh = 80\,\text{ft}^2 \cdot 2\,\text{ft} = \boxed{160\,\text{ft}^3}$$

Each bag holds $2\,\text{ft}^3$, so we needed $160 \div 2 = \boxed{80}$ bags.

Example 6. The base of a prism and of a pyramid is a right triangle with legs 4 cm and 3 cm. Find the volume of the prism and of the pyramid, if the height of the prism is 9 cm and the height of the pyramid is 6 cm.



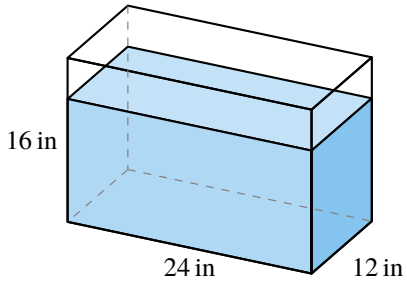
Solution: First we find the area of the base, a right triangle:

$$B = \frac{4\,\text{cm} \cdot 3\,\text{cm}}{2} = 6\,\text{cm}^2$$

$$V_{\text{prism}} = Bh = 6\,\text{cm}^2 \cdot 9\,\text{cm} = \boxed{54\,\text{cm}^3}$$

$$V_{\text{pyramid}} = \frac{Bh}{3} = \frac{6\,\text{cm}^2 \cdot 6\,\text{cm}}{3} = \boxed{12\,\text{cm}^3}$$

Example 7. A water tank is a rectangular prism, 24 inches long, 12 inches wide, and 16 inches high. It is filled with water to a height of 12 inches. How many cubic inches of water are in the tank? How many more cubic inches of water does it take to fill the tank?



Solution: The base is a 24 in by 12 in rectangle, so $B = 288\,\text{in}^2$. The water forms a prism with height 12 in:

$$V_{\text{water}} = 288\,\text{in}^2 \cdot 12\,\text{in} = \boxed{3456\,\text{in}^3}$$

The whole tank has volume $288\,\text{in}^2 \cdot 16\,\text{in} = 4608\,\text{in}^3$, so it takes $4608\,\text{in}^3 - 3456\,\text{in}^3 = \boxed{1152\,\text{in}^3}$ more water to fill it.

Example 8. A box is 2 feet long, 3 feet wide, and 1 foot high. Find its volume in cubic feet and in cubic inches.

Solution: In cubic feet,

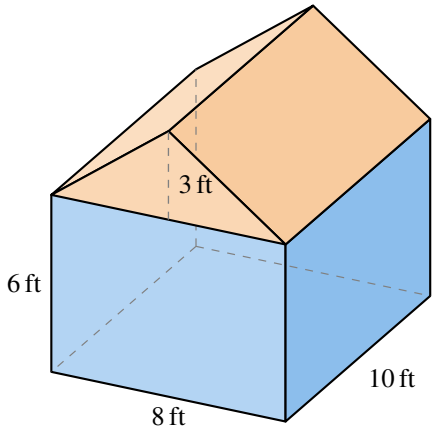
$$V = 2\,\text{ft} \cdot 3\,\text{ft} \cdot 1\,\text{ft} = \boxed{6\,\text{ft}^3}$$

For cubic inches, we first change each length to inches: the box is 24 in by 36 in by 12 in, so

$$V = 24\,\text{in} \cdot 36\,\text{in} \cdot 12\,\text{in} = \boxed{10\,368\,\text{in}^3}$$

Notice that we cannot simply multiply by 12: a cubic foot is a cube with edges 12 in, so $1\,\text{ft}^3 = 12 \cdot 12 \cdot 12\,\text{in}^3 = 1728\,\text{in}^3$, and indeed $6 \cdot 1728 = 10\,368$.

Example 9. A small shed is a rectangular prism 8 feet wide, 10 feet long, and 6 feet tall, with a roof in the shape of a triangular prism that is 3 feet higher than the walls. Find the volume of the shed, including the roof space.



Solution: We find the volumes of the two parts and add them. The walls form a rectangular prism:

$$V_{\text{walls}} = 8\,\text{ft} \cdot 10\,\text{ft} \cdot 6\,\text{ft} = 480\,\text{ft}^3$$

The roof is a prism lying on its side. Its base is the orange triangle at the front, with base 8 ft and height 3 ft, and the height of this prism is the length of the shed, 10 ft.

$$B = \frac{8\,\text{ft} \cdot 3\,\text{ft}}{2} = 12\,\text{ft}^2 \qquad V_{\text{roof}} = 12\,\text{ft}^2 \cdot 10\,\text{ft} = 120\,\text{ft}^3$$

The total volume is $480\,\text{ft}^3 + 120\,\text{ft}^3 = \boxed{600\,\text{ft}^3}$.

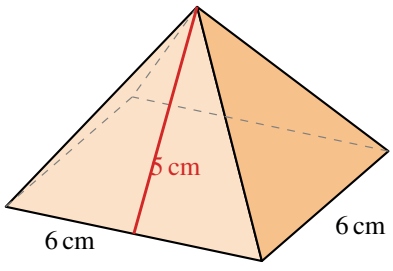
Example 10. A shipping crate is 6 feet long, 6 feet wide, and 8 feet high. How many boxes that are 2 feet by 3 feet by 4 feet fit into the crate?

Solution: The volume of the crate is $6 \cdot 6 \cdot 8 = 288\,\text{ft}^3$, and the volume of one box is $2 \cdot 3 \cdot 4 = 24\,\text{ft}^3$. If the boxes fill the crate completely with no gaps, we can fit

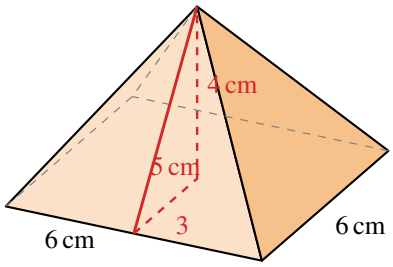
$$288 \div 24 = \boxed{12}$$

boxes. (Dividing volumes tells us the most boxes that could possibly fit. Here we can indeed stack 12 boxes: 3 along the length, 2 along the width, and 2 along the height.)

Example 11. A square-based pyramid has base edges 6 cm long. The **slant height** of the pyramid (the height of each triangular face, drawn in red) is 5 cm. Find the volume of the pyramid.



Solution: The base area is $B = 6\,\text{cm} \cdot 6\,\text{cm} = 36\,\text{cm}^2$, but the height of the pyramid is not given. The height h, half of a base edge (3 cm), and the slant height form a right triangle, with the slant height as its hypotenuse. By the Pythagorean theorem,

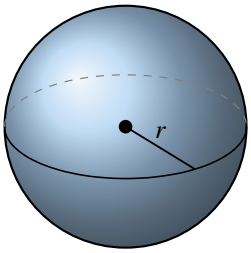


$$h^2 + 3^2 = 5^2 \qquad h^2 + 9 = 25 \qquad h^2 = 16 \qquad h = 4\,\text{cm}$$

$$V = \frac{Bh}{3} = \frac{36\,\text{cm}^2 \cdot 4\,\text{cm}}{3} = \boxed{48\,\text{cm}^3}$$

The Volume of a Sphere

Definition: A **sphere** is the set of all points in space that are the same distance from a fixed point. The fixed point is the **center** of the sphere, and the distance is the **radius** of the sphere.

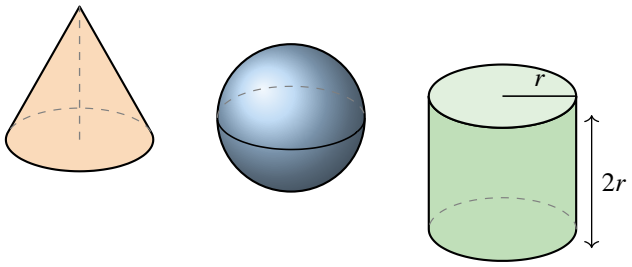


Theorem: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

A sphere is neither a solid nor a pointy object in our sense, so this formula has to be remembered on its own.

There is a famous picture of a cone, a sphere, and a cylinder with the same radius r and the same height $2r$: the sphere fits exactly into the cylinder, and so does the cone. Their volumes are

$$V_{\text{cone}} = \frac{\pi r^2 \cdot 2r}{3} = \frac{2}{3}\pi r^3$$
$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$
$$V_{\text{cylinder}} = \pi r^2 \cdot 2r = 2\pi r^3$$



$\frac{2}{3}\pi r^3$ $\frac{4}{3}\pi r^3$ $2\pi r^3$

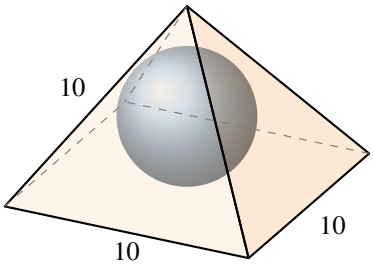
So the three volumes are in the ratio $1 : 2 : 3$. The cone and the sphere together have the same volume as the cylinder.

Example 12. Find the volume of a ball with a radius of 3 cm.

Solution:

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\,(3\,\text{cm})^3 = \frac{4 \cdot 27\pi}{3}\,\text{cm}^3 = \boxed{36\pi\,\text{cm}^3 \approx 113.097\,\text{cm}^3}$$

Example 13. All eight edges of a square-based pyramid are 10 units long. We put the largest possible sphere inside the pyramid: it touches the base and all four triangular faces. The volume of the sphere is what percent of the volume of the pyramid? Keep all digits on your calculator until the end, and round the percent to the nearest thousandth.



Solution: **The height of the pyramid.** We find the height in two steps: first the slant height s, then the height h. Each face of the pyramid is a triangle with three 10-unit sides. The slant height cuts it into two right triangles, with hypotenuse 10 and one leg 5 (half of a base edge):

$$s^2 + 5^2 = 10^2 \qquad s^2 + 25 = 100 \qquad s^2 = 75 \qquad s = \sqrt{75} = 5\sqrt{3}$$

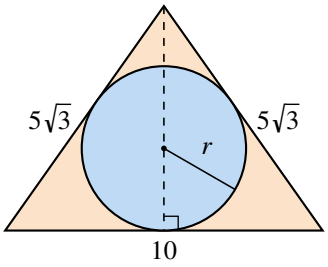
The height h, the slant height s, and the segment from the center of the base to the midpoint of a base edge (5 units) form a right triangle, with the slant height as its hypotenuse:

$$h^2 + 5^2 = \left(5\sqrt{3}\right)^2 \qquad h^2 + 25 = 75 \qquad h^2 = 50 \qquad h = \sqrt{50} = 5\sqrt{2} \approx 7.071$$

The volume of the pyramid. $B = 10 \cdot 10 = 100$, so

$$V_{\text{pyramid}} = \frac{Bh}{3} = \frac{100 \cdot 5\sqrt{2}}{3} = \frac{500\sqrt{2}}{3} \approx 235.702$$

The radius of the sphere. Now cut the pyramid in half through its tip and the midpoints of two opposite base edges. The cut is an isosceles triangle with base 10, two sides $5\sqrt{3}$, and height $5\sqrt{2}$ (see the picture). The cut goes through the center of the sphere, so the sphere shows up as the inscribed circle of this triangle, with the same radius r. For the inscribed circle, $A = \dfrac{Pr}{2}$, so $r = \dfrac{2A}{P}$:



$$A = \frac{10 \cdot 5\sqrt{2}}{2} = 25\sqrt{2} \qquad P = 10 + 5\sqrt{3} + 5\sqrt{3} = 10 + 10\sqrt{3}$$

$$r = \frac{2A}{P} = \frac{2 \cdot 25\sqrt{2}}{10 + 10\sqrt{3}} = \frac{5\sqrt{2}}{1 + \sqrt{3}} \approx 2.588$$

The volume of the sphere and the percent.

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3 \approx 72.624 \qquad \frac{V_{\text{sphere}}}{V_{\text{pyramid}}} \approx \frac{72.624}{235.702} \approx 0.30812 = \boxed{30.812\%}$$

So the sphere takes up about 30.812% of the pyramid.



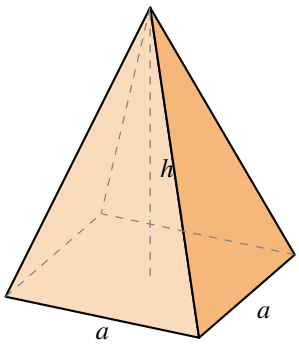
Practice Problems

Where an approximation is asked for, round to the nearest thousandth.

1. How many faces, edges, and vertices does each of these solids have: a cube, a square-based pyramid, and a prism whose base is a triangle?

2. What is the volume of a cube with edges 5 ft long?

3. Find the volume of the square-based pyramid shown on the picture, where $a = 4$ cm and $h = 6$ cm.



4. Suppose that the volume of a rectangular prism is 24 m^3. How many such prisms are possible if all three edges are whole numbers? (Prisms with the same three edges in a different order are not counted separately.)

5. Find the volume of a box that is 5 in long, 8 in wide, and 3 in high.

6. The base of a prism is a triangle with base 6 cm and height 4 cm. The height of the prism is 9 cm. Find the volume of the prism, and of a pyramid with the same base and a height of 9 cm.

7. A box is 3 feet long, 2 feet wide, and 2 feet high. Find its volume in cubic feet and in cubic inches.

8. A swimming pool is 20 meters long, 10 meters wide, and 2 meters deep. How many cubic meters of water does it hold when full? How many cubic meters of water are in it when the water level is 0.5 meters below the top?

9. A solid is made of a rectangular prism with an 8 in by 6 in base and a height of 4 in, with a rectangular pyramid on top of it. The pyramid has the same base and a height of 3 in. Find the volume of the whole solid.

10. The volume of a rectangular pyramid is 70 cm^3. Its base is a rectangle, 5 cm by 7 cm. What is the height of the pyramid?

11. Every edge of a cube is doubled. How many times larger does the volume become? Check with a cube of edge 1 and a cube of edge 2.

12. Rectangular prism A is 3 in long, 4 in wide, and 5 in high. Rectangular prism B is an enlargement of A by a factor of 2: each of its edges is twice as long. Find the volume of each prism. Is there a connection between the two volumes?

13. Square-based pyramid P has a base with sides of 3 cm and a height of 4 cm. Pyramid Q is similar to P, with every edge twice as long: its base has sides of 6 cm and its height is 8 cm. Find the volume of each pyramid. Is there a connection between the two volumes?

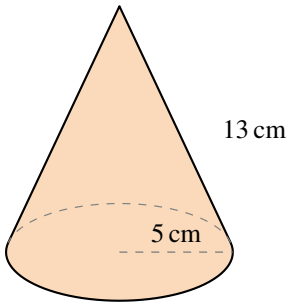
14. Find the volume of a cylinder with a base radius of 2 cm and a height of 7 cm. Give the exact answer and an approximation.

15. A cone has a base radius of 6 in and a height of 5 in. Find its volume.

16. A can of soup is a cylinder with a diameter of 8 cm and a height of 10 cm. How many cubic centimeters of soup does it hold when full?

17. A paper cup is shaped like a cone with a top radius of 3 cm and a height of 10 cm. One cubic centimeter is one milliliter. How many milliliters of water does the cup hold when full?

18. The base of a cone has a radius of 5 cm, and the slant edge of the cone (from the tip to the edge of the base) is 13 cm long. Find the volume of the cone.



19. A square-based pyramid has base edges 10 in long, and its slant height (the height of each triangular face) is 13 in. Find the volume of the pyramid.

20. Find the volume of a sphere with a radius of 6 in.

21. A basketball has a diameter of 24 cm. Find its volume.

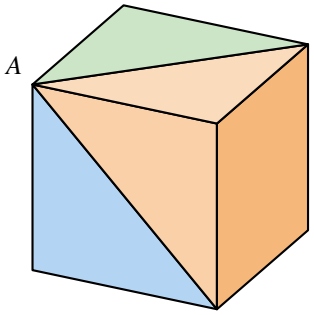
22. A sphere with a radius of 3 cm fits exactly into a cylinder: the radius of the cylinder is 3 cm, and its height is 6 cm. Find the volume of the sphere and of the cylinder. What fraction of the cylinder does the sphere fill? What percent is that?

23. A sphere fits exactly into a cube with 10 cm edges: it touches all six faces of the cube. The volume of the sphere is what percent of the volume of the cube?



Enrichment

A cube with 6 cm edges can be cut into three pyramids, shown on the picture in three colors. All three pyramids have their tip at the corner A of the cube. Each pyramid has a face of the cube as its base. Find the volume of each pyramid in two ways: with the formula $V = \frac{Bh}{3}$, and as a part of the cube. What does this tell us about the formula?





Answers

Practice Problems

1. Cube: 6 faces, 12 edges, 8 vertices. Square-based pyramid: 5 faces, 8 edges, 5 vertices. Triangular prism: 5 faces, 9 edges, 6 vertices.
2. $V = 5\,\text{ft} \cdot 5\,\text{ft} \cdot 5\,\text{ft} = 125\,\text{ft}^3$
3. $B = 16\,\text{cm}^2,\ V = \dfrac{16\,\text{cm}^2 \cdot 6\,\text{cm}}{3} = 32\,\text{cm}^3$
4. 6 possibilities: $1 \times 1 \times 24,\ 1 \times 2 \times 12,\ 1 \times 3 \times 8,\ 1 \times 4 \times 6,\ 2 \times 2 \times 6,\ 2 \times 3 \times 4$
5. $V = 5 \cdot 8 \cdot 3 = 120\,\text{in}^3$
6. $B = \dfrac{6 \cdot 4}{2} = 12\,\text{cm}^2$. Prism: $V = 12 \cdot 9 = 108\,\text{cm}^3$. Pyramid: $V = \dfrac{12 \cdot 9}{3} = 36\,\text{cm}^3$.
7. $V = 3 \cdot 2 \cdot 2 = 12\,\text{ft}^3$. In inches, the box is 36 by 24 by 24, so $V = 36 \cdot 24 \cdot 24 = 20\,736\,\text{in}^3$ (which is $12 \cdot 1728$).
8. Full: $20 \cdot 10 \cdot 2 = 400\,\text{m}^3$. With the water 1.5 meters deep: $20 \cdot 10 \cdot 1.5 = 300\,\text{m}^3$.
9. Prism: $48 \cdot 4 = 192\,\text{in}^3$. Pyramid: $\dfrac{48 \cdot 3}{3} = 48\,\text{in}^3$. Total: $240\,\text{in}^3$.
10. 6 cm ($B = 35\,\text{cm}^2$, so $\dfrac{35h}{3} = 70,\ 35h = 210,\ h = 6$)
11. $1^3 = 1$ and $2^3 = 8$: the volume becomes 8 times larger. (Doubling all three dimensions multiplies the volume by $2 \cdot 2 \cdot 2$.)
12. Prism A: $V = 3 \cdot 4 \cdot 5 = 60\,\text{in}^3$. Prism B is 6 in by 8 in by 10 in, so $V = 6 \cdot 8 \cdot 10 = 480\,\text{in}^3$. Since $480 = 8 \cdot 60$, the volume of the larger prism is 8 times the volume of the smaller one: each of the three dimensions doubled, and $2 \cdot 2 \cdot 2 = 8$.
13. Pyramid P: $B = 9\,\text{cm}^2$, $V = \dfrac{9 \cdot 4}{3} = 12\,\text{cm}^3$. Pyramid Q: $B = 36\,\text{cm}^2$, $V = \dfrac{36 \cdot 8}{3} = 96\,\text{cm}^3$. Since $96 = 8 \cdot 12$, the volume of the larger pyramid is again 8 times the volume of the smaller one.
14. $B = 4\pi\,\text{cm}^2,\ V = 28\pi\,\text{cm}^3 \approx 87.965\,\text{cm}^3$
15. $B = 36\pi\,\text{in}^2,\ V = \dfrac{36\pi \cdot 5}{3} = 60\pi\,\text{in}^3 \approx 188.496\,\text{in}^3$
16. The radius is 4 cm, $B = 16\pi\,\text{cm}^2,\ V = 160\pi\,\text{cm}^3 \approx 502.655\,\text{cm}^3$
17. $V = \dfrac{9\pi \cdot 10}{3} = 30\pi\,\text{cm}^3 \approx 94.248\,\text{cm}^3$, so about 94.248 milliliters
18. The height, the radius, and the slant edge form a right triangle: $h^2 + 5^2 = 13^2$, so $h^2 = 144$ and $h = 12\,\text{cm}$. $V = \dfrac{25\pi \cdot 12}{3} = 100\pi\,\text{cm}^3 \approx 314.159\,\text{cm}^3$
19. The height, half of a base edge, and the slant height form a right triangle: $h^2 + 5^2 = 13^2$, so $h = 12\,\text{in}$. $V = \dfrac{100 \cdot 12}{3} = 400\,\text{in}^3$
20. $V = \dfrac{4}{3}\pi \cdot 6^3 = 288\pi\,\text{in}^3 \approx 904.779\,\text{in}^3$
21. The radius is 12 cm, $V = \dfrac{4}{3}\pi \cdot 12^3 = 2304\pi\,\text{cm}^3 \approx 7238.229\,\text{cm}^3$

22. Sphere: $36\pi\,\text{cm}^3 \approx 113.097\,\text{cm}^3$. Cylinder: $\pi \cdot 3^2 \cdot 6 = 54\pi\,\text{cm}^3 \approx 169.646\,\text{cm}^3$. The sphere fills $\frac{36\pi}{54\pi} = \frac{2}{3}$ of the cylinder, about 66.667%. (Archimedes was so proud of this discovery that he asked for a sphere in a cylinder on his tombstone.)

23. The radius of the sphere is 5 cm: $V_{\text{sphere}} = \frac{4}{3}\pi \cdot 125 = \frac{500\pi}{3}\,\text{cm}^3 \approx 523.599\,\text{cm}^3$, and $V_{\text{cube}} = 1000\,\text{cm}^3$. The sphere is about 52.360% of the cube.

Enrichment

Each pyramid has a 6 cm by 6 cm face of the cube as its base, so $B = 36\,\text{cm}^2$. Its tip A is a corner of the opposite face, so its height is an edge of the cube, 6 cm. With the formula, $V = \frac{36 \cdot 6}{3} = 72\,\text{cm}^3$. The three pyramids are identical, and together they fill the cube, so each of them is one third of the cube: $\frac{6 \cdot 6 \cdot 6}{3} = \frac{216}{3} = 72\,\text{cm}^3$. This shows where the "over 3" in the formula for pointy objects comes from: three such pyramids fill the solid (the cube) with the same base and height.

