Sample Problems

Compute each of the following limits. Show all steps, using correct notation.

1.) $\lim\limits_{x \to \infty} 2^x$

2.) $\lim\limits_{x \to -\infty} 2^x$

3.) $\lim\limits_{x \to \infty} \left(\frac{2}{3}\right)^x$

4.) $\lim\limits_{x \to -\infty} \left(\frac{2}{3}\right)^x$

5.) $\lim\limits_{x \to \infty} \frac{2^{x+3}}{3^{x+1}}$

6.) $\lim\limits_{x \to -\infty} \frac{2^{x+3}}{3^{x+1}}$

7.) $\lim\limits_{x \to \infty} \frac{2^{2x+1}}{3^{x-1}}$

8.) $\lim\limits_{x \to -\infty} \frac{2^{2x+1}}{3^{x-1}}$

9.) $\lim\limits_{x \to \infty} \left(3^{x+1} - 3^x\right)$

10.) $\lim\limits_{x \to \infty} \frac{3\sqrt{x}+2}{5\sqrt{x}+1}$

11.) $\lim\limits_{x \to \infty} \left(\sqrt{2x-1} - \sqrt{2x}\right)$

12.) $\lim\limits_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x+1} - \sqrt{2x}}$

13.) $\lim\limits_{x \to \infty} x\left(\frac{1}{5} - \frac{1}{5 - \frac{1}{x}}\right)$

14.) $\lim\limits_{x \to \infty} \frac{\cos x}{x}$

15.) $\lim\limits_{x \to \infty} \frac{x + \frac{1}{x}}{x - \frac{1}{x}}$

16.) $\lim\limits_{x \to \infty} \frac{2^x + 2^{-x}}{2^x - 2^{-x}}$

17.) $\lim\limits_{x \to -\infty} \left(\sqrt{3x-1} - \sqrt{3x+1}\right)$

Practice Problems

Compute each of the following limits. Show all steps, using correct notation.

1.) $\lim\limits_{x \to \infty} \frac{2^{3x-1}}{5^{x-1}}$

2.) $\lim\limits_{x \to -\infty} \frac{2^{3x-1}}{5^{x-1}}$

3.) $\lim\limits_{x \to \infty} \frac{2^{2x+3}}{5^{x-1}}$

4.) $\lim\limits_{x \to -\infty} \frac{2^{2x+3}}{5^{x-1}}$

5.) $\lim\limits_{x \to \infty} \frac{2^{2x+3}}{4^{x-1}}$

6.) $\lim\limits_{x \to -\infty} \frac{2^{2x+3}}{4^{x-1}}$

7.) $\lim\limits_{x \to \infty} \frac{2^{x+3} \cdot 3^{x-1}}{7^{x-2}}$

8.) $\lim\limits_{x \to \infty} \frac{2^{2x+3} \cdot 3^{x-1}}{7^{x-2}}$

9.) $\lim\limits_{x \to \infty} \left(\sqrt{x+1} - \sqrt{x}\right)$

10.) $\lim\limits_{x \to \infty} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$

11.) $\lim\limits_{x \to \infty} \frac{x}{\sqrt{x-1} + \sqrt{x+1}}$

12.) $\lim\limits_{x \to -\infty} \frac{\sqrt{x}}{\sqrt{x+1} - \sqrt{2x}}$

13.) $\lim\limits_{x \to \infty} x\left(\frac{1}{a} - \frac{1}{a - \frac{1}{x}}\right)$

14.) $\lim\limits_{x \to \infty} \frac{0.5^x + 0.5^{-x}}{0.5^x - 0.5^{-x}}$

15.) $\lim\limits_{x \to \infty} \frac{\sin x - \cos x}{\sqrt{x^2+1}}$

Sample Problems - Answers

1.) ∞ 2.) 0 3.) 0 4.) ∞ 5.) 0 6.) ∞ 7.) ∞ 8.) 0 9.) ∞

10.) $\dfrac{3}{5}$ 11.) 0 12.) $\dfrac{1}{1-\sqrt{2}} = -1-\sqrt{2}$ 13.) $-\dfrac{1}{25}$ 14.) 0 15.) 1 16.) 1

17.) undefined

Practice Problems - Answers

1.) ∞ 2.) 0 3.) 0 4.) ∞ 5.) 32 6.) 32 7.) 0 8.) ∞ 9.) 0

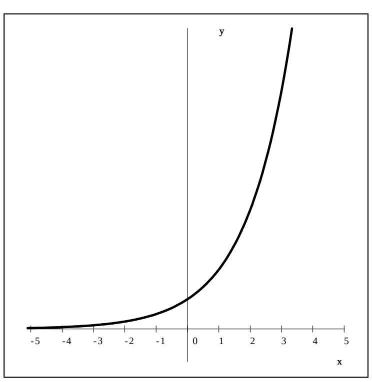
10.) 0 11.) ∞ 12.) undefined 13.) $-\dfrac{1}{a^2}$ 14.) -1 15.) 0

Sample Problems - Solutions

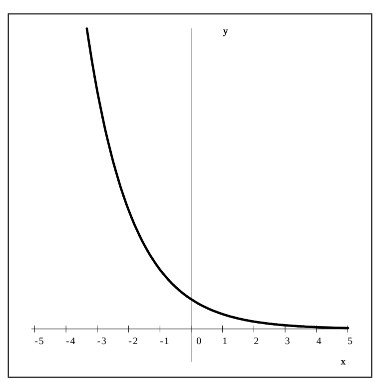
Let $a > 0$. Then the limit of the exponential function $f(x) = a^x$ is as follows.

Case 1. If $a > 1$, then $\lim\limits_{x\to\infty} a^x = \infty$ and $\lim\limits_{x\to-\infty} a^x = 0$

Case 2. If $0 < a < 1$, then $\lim\limits_{x\to\infty} a^x = 0$ and $\lim\limits_{x\to-\infty} a^x = \infty$



$a > 1$



$0 < a < 1$

1.) $\lim\limits_{x\to\infty} 2^x$ and 2.) $\lim\limits_{x\to-\infty} 2^x$

Solution: Since $2 > 1$, these limits are ∞ and 0, i.e. $\lim\limits_{x\to\infty} 2^x = \infty$ and $\lim\limits_{x\to-\infty} 2^x = 0$.

3.) $\lim_{x\to\infty}\left(\frac{2}{3}\right)^x$ and 4.) $\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x$

Solution: Since $\frac{2}{3}<1$, these limits are 0 and ∞, i.e. $\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0$ and $\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x=\infty$.

5.) $\lim_{x\to\infty}\frac{2^{x+3}}{3^{x+1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x.

$$\frac{2^{x+3}}{3^{x+1}}=\frac{2^x\cdot 2^3}{3^x\cdot 3^1}=\frac{2^x\cdot 8}{3^x\cdot 3}=\frac{8}{3}\left(\frac{2}{3}\right)^x$$

$$\text{Thus}\quad \lim_{x\to\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to\infty}\frac{8}{3}\left(\frac{2}{3}\right)^x=\frac{8}{3}\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0 \qquad \text{since } \frac{2}{3}<1$$

6.) $\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}$

Solution: $\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to-\infty}\frac{8}{3}\left(\frac{2}{3}\right)^x=\frac{8}{3}\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x=\infty$

7.) $\lim_{x\to\infty}\frac{2^{2x+1}}{3^{x-1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x.

$$\frac{2^{2x+1}}{3^{x-1}}=\frac{2^{2x}\cdot 2^1}{\frac{3^x}{3^1}}=\frac{\left(2^2\right)^x\cdot 2}{3^x\cdot\frac{1}{3}}=\frac{4^x\cdot 6}{3^x}=6\left(\frac{4}{3}\right)^x$$

$$\text{Thus}\quad \lim_{x\to\infty}\frac{2^{2x+1}}{3^{x-1}}=\lim_{x\to\infty}6\left(\frac{4}{3}\right)^x=6\lim_{x\to\infty}\left(\frac{4}{3}\right)^x=\infty \quad \text{since } \frac{4}{3}>1$$

8.) $\lim_{x\to-\infty}\frac{2^{2x+1}}{3^{x-1}}$

Solution: $\lim_{x\to-\infty}\frac{2^{2x+1}}{3^{x-1}}=\lim_{x\to-\infty}6\left(\frac{4}{3}\right)^x=6\lim_{x\to-\infty}\left(\frac{4}{3}\right)^x=0$

9.) $\lim_{x\to\infty}\left(3^{x+1}-3^x\right)$

Solution: $\lim_{x\to\infty}\left(3^{x+1}-3^x\right)=\lim_{x\to\infty}\left(3^x\cdot 3-3^x\right)=\lim_{x\to\infty}\left(3\cdot 3^x-3^x\right)=\lim_{x\to\infty}\left(2\cdot 3^x\right)=2\lim_{x\to\infty}3^x=\infty$

10.) $\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}$

Solution: Since $\lim_{x\to\infty}\sqrt{x}=\infty$, clearly $\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$. We will use this fact; we factor out $\sqrt{x}$ from both numerator and denominator.

$$\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}=\lim_{x\to\infty}\frac{\sqrt{x}\left(3+\frac{2}{\sqrt{x}}\right)}{\sqrt{x}\left(5+\frac{1}{\sqrt{x}}\right)}=\lim_{x\to\infty}\frac{3+\frac{2}{\sqrt{x}}}{5+\frac{1}{\sqrt{x}}}=\frac{3}{5}$$

11.) $\displaystyle\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right)$

Solution: We will transform this expression by multiplying it by 1, written as a fraction with numerator and denominator both being the conjugate of the expression.

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right) &= \lim_{x\to\infty}\frac{\sqrt{2x-1}-\sqrt{2x}}{1}\cdot\frac{\sqrt{2x-1}+\sqrt{2x}}{\sqrt{2x-1}+\sqrt{2x}}\\ &= \lim_{x\to\infty}\frac{(2x-1)-(2x)}{\sqrt{2x-1}+\sqrt{2x}}=\lim_{x\to\infty}\frac{-1}{\sqrt{2x-1}+\sqrt{2x}}=0\end{aligned}$$

12.) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

Solution: We factor out $\sqrt{x}$ from both numerator and denominator.

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}} &= \lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}\left(\dfrac{\sqrt{x+1}}{\sqrt{x}}-\sqrt{2}\right)}=\lim_{x\to\infty}\frac{1}{\sqrt{\dfrac{x+1}{x}}-\sqrt{2}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{1+\dfrac{1}{x}}-\sqrt{2}}=\frac{1}{1-\sqrt{2}}=\frac{1}{1-\sqrt{2}}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}}=\frac{1+\sqrt{2}}{-1}\\ &= -1-\sqrt{2}\end{aligned}$$

13.) $\displaystyle\lim_{x\to\infty}x\left(\frac{1}{5}-\frac{1}{5-\dfrac{1}{x}}\right)$

Solution: We just need to simplify the complex fraction. As it turns out, this problem boils down to a type we have already seen.

$$\begin{aligned}x\left(\frac{1}{5}-\frac{1}{5-\dfrac{1}{x}}\right) &= x\left(\frac{1}{5}-\frac{1}{\dfrac{5x-1}{x}}\right)=x\left(\frac{1}{5}-\frac{x}{5x-1}\right)=x\left(\frac{(5x-1)-5x}{5(5x-1)}\right)\\ &= x\left(\frac{5x-1-5x}{5(5x-1)}\right)=x\frac{-1}{25x-5}=\frac{-x}{25x-5}\end{aligned}$$

Thus

$$\lim_{x\to\infty}x\left(\frac{1}{5}-\frac{1}{5-\dfrac{1}{x}}\right)=\lim_{x\to\infty}\frac{-x}{25x-5}=\lim_{x\to\infty}\frac{x(-1)}{x\left(25-\dfrac{5}{x}\right)}=-\frac{1}{25}$$

14.) $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}$

Solution: This problem can be solved by the sandwich principle. Consider the limits $\displaystyle\lim_{x\to\infty}\frac{1}{x}$ and $\displaystyle\lim_{x\to\infty}\left(-\frac{1}{x}\right)$. These limits are both zero. Furthermore, since

$$\begin{array}{rcll} -1 & \leq & \cos x \leq 1 & \text{for all } x\text{, we also have} \\ -\dfrac{1}{x} & \leq & \dfrac{\cos x}{x} \leq \dfrac{1}{x} & \text{for all positive } x \end{array}$$

Our function $f(x)=\dfrac{\cos x}{x}$ is 'locked' between $g(x)=\dfrac{1}{x}$ and $h(x)=-\dfrac{1}{x}$. Since these both approach zero, so must the function $f(x)=\dfrac{\cos x}{x}$. Thus $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}=0$.

15.) $\displaystyle\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}$

Solution: We will factor out x from both numerator and denominator, and use the fact that $\displaystyle\lim_{x\to\infty}\frac{1}{x^k}=0$ for all positive integers k..

$$\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x^2}\right)}{x\left(1-\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}=1$$

16.) $\displaystyle\lim_{x\to\infty}\frac{2^x+2^{-x}}{2^x-2^{-x}}$

Solution: First, $\displaystyle\lim_{x\to\infty}2^x=\infty$ (and so 2^x is large) and $\displaystyle\lim_{x\to\infty}2^{-x}=0$ (and so 2^{-x} is small). With that in mind, this limit is similar to $\displaystyle\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}$. The solution also will be similar. We will factor out 2^x from both numerator and denominator.

$$\lim_{x\to\infty}\frac{2^x+2^{-x}}{2^x-2^{-x}}=\lim_{x\to\infty}\frac{2^x+\frac{1}{2^x}}{2^x-\frac{1}{2^x}}=\lim_{x\to\infty}\frac{2^x\left(1+\frac{1}{(2^x)^2}\right)}{2^x\left(1-\frac{1}{(2^x)^2}\right)}=\lim_{x\to\infty}\frac{1+\frac{1}{2^{2x}}}{1-\frac{1}{2^{2x}}}=1$$

17.) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{3x-1}-\sqrt{3x+1}\right)$

Solution: When $x\to-\infty$, then we may assume it is negative. Then the expressions under the square root are negative and the function is not defined. Thus, there is no limit at negative infinity. The answer is: undefined.