

1. Find all critical points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Solution:

$$\begin{aligned} f_x(x, y) &= 2x - 2 = 0 \implies x = 1 \\ f_y(x, y) &= 2y - 6 \implies y = 3 \end{aligned}$$

The only critical point is $(1, 3, 4)$.

2. Find the minimal distance from the origin to the plane $x - y + 2z = 3$.

Solution: $d = \sqrt{x^2 + y^2 + \left(\frac{3 - x + y}{2}\right)^2}$. Let $f(x, y) = d^2 = x^2 + y^2 + \left(\frac{3 - x + y}{2}\right)^2$. Instead of minimizing the distance, we minimize the square of the distance.

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 2\left(\frac{3 - x + y}{2}\right)\left(-\frac{1}{2}\right) = \frac{5}{2}x - \frac{y}{2} - \frac{3}{2} = 0 \implies 5x - y = 3 \\ \frac{\partial f}{\partial y} &= 2y + 2\left(\frac{3 - x + y}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}x + \frac{5}{2}y + \frac{3}{2} = 0 \implies -x + 5y = -3 \end{aligned}$$

We solve the system $\begin{cases} 5x - y = 3 \\ -x + 5y = -3 \end{cases}$ and obtain $x = \frac{1}{2}$, $y = -\frac{1}{2}$. Then $z = \frac{3 - \frac{1}{2} - (-\frac{1}{2})}{2} = 1$, so the closest point is $\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$, and the minimal distance is $d = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{6}}{2}$.

3. Find the local maximum or minimum values of $f(x, y) = x^2 - 2x + \frac{y^2}{4}$.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x - 2 = 0 \implies x = 1 \\ \frac{\partial f}{\partial y} &= \frac{2y}{4} = \frac{y}{2} = 0 \implies y = 0 \end{aligned}$$

The only critical point is $(1, 0)$, with $f(1, 0) = -1$.

$f(x, y) = (x - 1)^2 + \frac{y^2}{4} - 1 \geq -1$ for all (x, y) in domain of f , and so $(1, 0, -1)$ is a global minimum for f .

4. Find local extrema for $z = f(x, y) = -x^2 - y^2 + 6x + 8y - 21$.

Solution:

$$\begin{aligned} f(x, y) &= -x^2 - y^2 + 6x + 8y - 21 = -(x^2 - 6x + y^2 - 8y + 21) = -((x - 3)^2 - 9 + (y - 4)^2 - 16 + 21) \\ &= -(x - 3)^2 - (y - 4)^2 + 4 \quad (3, 4, 4) \text{ is a local and global maximum} \end{aligned}$$

5. a) Apply the maximum-minimum test to $f(x, y) = x^2 + 3xy + y^2 + 16$

b) Determine whether $(0, 0)$ is a maximum-point, a minimum-point, or neither for $g(x, y) = 3x^2 - 5xy + 3y^2$.

Solution: a) $A = 1, B = \frac{3}{2}, C = 1$ $AC - B^2 = 1 \cdot 1 - \left(\frac{3}{2}\right)^2 = -\frac{5}{4}$ thus $(0, 0)$ is a saddle

point. Also, $f(x, y) = \left(x + \frac{3}{2}y\right)^2 - \frac{9}{4}y^2$

b) $A = 3, B = -\frac{5}{2}, C = 3$ $AC - B^2 = 3 \cdot 3 - \left(-\frac{5}{2}\right)^2 = \frac{11}{4}$. f has a minimum at $(0, 0)$.

6. Find the critical points of each function and classify them as local maxima, minima, or neither.

a) $f(x, y) = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$

b) $f(x, y) = x^4 + y^4 - 4xy + 1$

c) $f(x, y) = e^{(1-x^2-y^2)}$

d) $f(x, y) = (x^2 - y^2)e^{-\frac{x^2+y^2}{2}}$

Solution:

a) $f(x, y) = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$f_x(x, y) = -\frac{2x}{a^2} \quad (0, 0) \text{ is the only critical point}$$

$$f_y(x, y) = \frac{2y}{b^2}$$

Second derivatives:

$$f_{xx}(x, y) = -\frac{2}{a^2} \quad A = f_{xx}(0, 0) = -\frac{2}{a^2}$$

$$f_{xy}(x, y) = 0 \quad B = f_{xy}(0, 0) = 0$$

$$f_{yy}(x, y) = \frac{2}{b^2} \quad C = f_{yy}(0, 0) = \frac{2}{b^2}$$

Second derivative test:

$$A < 0 \quad \text{and} \quad AC - B^2 = \left(-\frac{2}{a^2}\right)\left(\frac{2}{b^2}\right) - 0^2 = -\frac{4}{a^2b^2} < 0$$

$(0, 0, 0)$ is a saddle point of f .

b) $f(x, y) = x^4 + y^4 - 4xy + 1$

$$f_x(x, y) = 4x^3 - 4y = 0 \quad x^3 = y$$

$$f_y(x, y) = 4y^3 - 4x = 0 \quad y^3 = x$$

We solve the system and find three critical points: $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Second derivatives:

$$f_{xx}(x, y) = 12x^2 \quad f_{xy}(x, y) = -4 \quad f_{yy}(x, y) = 12y^2$$

Second derivative test:

critical point	A	B	C	$AC - B^2$	Type
$(0, 0)$	0	-4	0	-16	saddle point
$(1, 1)$	12	-4	12	128	minimum
$(-1, -1)$	12	-4	12	128	minimum

Thus $(0, 0, 1)$ is a saddle point, and $(1, 1, -1)$ and $(-1, -1, -1)$ are local minima.

c) $f(x, y) = e^{(1-x^2-y^2)}$

Solution:

$$\begin{aligned} f_x(x, y) &= -2xe^{(1-x^2-y^2)} & (0, 0) \text{ is the only critical point} \\ f_y(x, y) &= -2ye^{(1-x^2-y^2)} \end{aligned}$$

Second derivatives:

$$\begin{aligned} f_{xx}(x, y) &= -2 \left(e^{(1-x^2-y^2)} + xe^{(1-x^2-y^2)}(-2x) \right) = 2(2x^2 - 1)e^{(1-x^2-y^2)} \\ A &= f_{xx}(0, 0) = -2e \\ f_{xy}(x, y) &= -2xe^{(1-x^2-y^2)}(-2y) = 4xye^{(1-x^2-y^2)} \\ B &= f_{xy}(0, 0) = 0 \\ f_{yy}(x, y) &= -2 \left(e^{(1-x^2-y^2)} + ye^{(1-x^2-y^2)}(-2y) \right) = 2(2y^2 - 1)e^{(1-x^2-y^2)} \\ C &= f_{yy}(0, 0) = -2e \end{aligned}$$

Second derivative test:

$$A < 0 \quad \text{and} \quad AC - B^2 = (-2e)(-2e) - 0^2 = 4e^2 > 0$$

Thus f has a local maximum: $(0, 0, e)$

d) $f(x, y) = (x^2 - y^2)e^{\frac{-x^2-y^2}{2}}$

$$\begin{aligned} f_x(x, y) &= x(y^2 - x^2 + 2)e^{\frac{-x^2-y^2}{2}} = 0 & x(y^2 - x^2 + 2) &= 0 \\ f_y(x, y) &= -y(x^2 - y^2 + 2)e^{\frac{-x^2-y^2}{2}} = 0 & -y(x^2 - y^2 + 2) &= 0 \end{aligned}$$

We solve the system

$$\begin{aligned} x(y^2 - x^2 + 2) &= 0 \\ -y(x^2 - y^2 + 2) &= 0 \end{aligned}$$

and obtain $(0, 0)$, $(0, \sqrt{2})$, $(0, -\sqrt{2})$, $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

Second derivatives:

$$\begin{aligned} f_{xx}(x, y) &= (y^2 - 5x^2 + x^4 - x^2y^2 + 2)e^{\frac{-x^2-y^2}{2}} \\ f_{xy}(x, y) &= xy(x^2 - y^2)e^{\frac{-x^2-y^2}{2}} \\ f_{yy}(x, y) &= (-x^2 + 5y^2 - y^4 + x^2y^2 - 2)e^{\frac{-x^2-y^2}{2}} \end{aligned}$$

critical point	A	B	C	$AC - B^2$	Type
$(0, 0, 0)$	2	0	-2	-4	saddle point
$(\sqrt{2}, 0, 2/e)$	$-4/e$	0	$-4/e$	$16/e^2$	maximum
$(-\sqrt{2}, 0, 2/e)$	$-4/e$	0	$-4/e$	$16/e^2$	maximum
$(0, \sqrt{2}, -2/e)$	$4/e$	0	$4/e$	$16/e^2$	minimum
$(0, -\sqrt{2}, -2/e)$	$4/e$	0	$4/e$	$16/e^2$	minimum

7. Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Solution: $f(x, y) = d^2 = (x - 1)^2 + y^2 + (4 - x - 2y + 2)^2 = (x - 1)^2 + y^2 + (6 - x - 2y)^2$

$$f_x(x, y) = 2(x - 1) + 2(6 - x - 2y)(-1) = 4x + 4y - 14 = 0 \quad 2x + 2y = 7$$

$$f_y(x, y) = 2y + 2(6 - x - 2y)(-2) = 4x + 10y - 24 \quad 2x + 5y = 12$$

We solve the system and obtain $x = \frac{11}{6}$ and $y = \frac{5}{3}$. Then $z = -\frac{7}{6}$ and the shortest distance is

$$d = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{7}{6} + 2\right)^2} = \frac{5}{6}\sqrt{6}.$$

8. An open rectangular box is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution: Let x, y , and z denote the dimensions, z being the vertical side. Notice that $x, y, z > 0$

$$A = xy + 2xz + 2yz \implies z = \frac{A - xy}{2x + 2y}$$

$$V(x, y) = xyz = xy \left(\frac{A - xy}{2x + 2y} \right) = \frac{1}{2} \left(\frac{Axy - x^2y^2}{x + y} \right)$$

$$V_x(x, y) = \frac{1}{4} \frac{y^2(A - 2xy - x^2)}{(x + y)^2} \quad A - 2xy - x^2 = 0$$

$$V_y(x, y) = \frac{1}{4} \frac{x^2(A - 2xy - y^2)}{(x + y)^2} \quad A - 2xy - y^2 = 0$$

We get that $x = y$ and $A = 3x^2$ and so $x = \sqrt{\frac{A}{3}} = \sqrt{4} = 2$. Thus $x = y = 2$ and $z = 1$. The maximum volume is 4 m^3 .

9. A surfboard company has developed the yearly profit equation

$$P(x, y) = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$$

where x is the number (in thousands) of standard surfboards produced per year, and y is the number (in thousands) of competition surfboards produced each year, and P is the profit (in thousands of dollars). How many of each type of board should be produced per year to realize a maximal profit? What is the maximal profit?

$$\begin{aligned} \frac{\partial P}{\partial x} &= -44x + 22y + 110 \\ \frac{\partial P}{\partial y} &= 22x - 22y - 44 \end{aligned}$$

$$\begin{aligned}
 0 &= -2x + y + 5 \\
 0 &= x - y - 2 & x = 3, y = 1 \\
 P(3, 1) &= 120
 \end{aligned}$$

And so \$120 000 is the maximum profit when 3000 standard and 1000 competition boards are produced.

10. A department store sells two brands of inexpensive calculators. The store pays \$6 for each brand *A* calculator and \$8 for each brand *B* calculator. The research department has estimated the following weekly demand for these two competitive products.

$$x = 116 - 30p + 20q \quad \text{and} \quad y = 144 + 16p - 24q$$

where p is the selling price for brand *A* and q is the selling price for brand *B*.

a) Determine the demands x and y when $p = \$10$ and $q = \$12$, when $p = q = \$11$.

b) How should the store price each calculator to maximize weekly profit? What is the maximum weekly profit?

$$\text{Solution: } x = 116 - 30(10) + 20(12) = 56 \quad y = 144 + 16(10) - 24(12) = 16$$

$$x = 116 - 30(11) + 20(11) = 6 \quad \text{and} \quad y = 144 + 16(11) - 24(11) = 56$$

$$\begin{aligned}
 P(p, q) &= (p - 6)x + (q - 8)y \\
 &= (p - 6)(116 - 30p + 20q) + (q - 8)(144 + 16p - 24q)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial P}{\partial p} &= 12(3q - 5p + 14) \\
 \frac{\partial P}{\partial q} &= -12(4q - 3p - 18)
 \end{aligned}$$

$$\begin{aligned}
 0 &= 12(3q - 5p + 14) \\
 0 &= -12(4q - 3p - 18) & p = 10, q = 12 \\
 P(10, 12) &= 288
 \end{aligned}$$