Sample Problems

Compute each of the following integrals. Please note that $\arcsin x$ is the same as $\sin^{-1} x$.

1. $\displaystyle\int e^{-4x}\, dx$

2. $\displaystyle\int_0^8 e^{-4x}\, dx$

3. $\displaystyle\int \left(x^2-2\right)\left(x^3-6x\right)^{207}\, dx$

4. $\displaystyle\int e^{\cos x}\sin x\, dx$

5. $\displaystyle\int_0^{\pi/2} e^{\cos x}\sin x\, dx$

6. $\displaystyle\int \frac{3x}{\left(x^2+1\right)^7}\, dx$

7. $\displaystyle\int \frac{12x^3}{3x^4+1}\, dx$

8. $\displaystyle\int (-3x+4)\, e^{-3x^2+8x}\, dx$

9. $\displaystyle\int_0^2 (-3x+4)\, e^{-3x^2+8x}\, dx$

10. $\displaystyle\int \frac{x+5}{x^2+1}\, dx$

11. $\displaystyle\int \frac{e^x}{e^x+1}\, dx$

12. $\displaystyle\int \cos x \sin^5 x\, dx$

13. $\displaystyle\int \frac{x}{\sqrt{x+1}}\, dx$

14. $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\, dx$

15. $\displaystyle\int_0^3 \frac{e^x-e^{-x}}{e^x+e^{-x}}\, dx$

16. $\displaystyle\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\, dx$

17. $\displaystyle\int_0^1 \frac{x+1}{x^2+1}\, dx$

18. $\displaystyle\int_0^\infty xe^{-3x^2}\, dx$

19. $\displaystyle\int_0^\infty 3e^{-3x}\, dx$

20. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

21. $\displaystyle\int \frac{1}{x^2+9}\, dx$

22. $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$

Sample Problems - Answers

1.) $-\frac{1}{4}e^{-4x}+C$ 2.) $\frac{1}{4}-\frac{1}{4}e^{-32}$ 3.) $\frac{1}{624}\left(x^3-6x\right)^{208}+C$ 4.) $-e^{\cos x}+C$

5.) $e-1$ 6.) $-\frac{1}{4\left(x^2+1\right)^6}+C$ 7.) $\ln\left(3x^4+1\right)+C$ 8.) $\frac{1}{2}e^{-3x^2+8x}+C$

9.) $\frac{1}{2}e^4-\frac{1}{2}$ 10.) $5\arctan x+\frac{1}{2}\ln\left(x^2+1\right)+C$ 11.) $\ln\left(e^x+1\right)+C$ 12.) $\frac{1}{6}\sin^6 x+C$

13.) $\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+C$ 14.) 36 15.) $\ln\left(e^3+\frac{1}{e^3}\right)-\ln 2$

16.) $2\ln\left(\sqrt{x}+1\right)+C$ 17.) $\frac{\pi}{4}+\frac{\ln 2}{2}$ 18.) $\frac{1}{6}$ 19.) 1 20.) $-2\cos\sqrt{x}+C$

21.) $\frac{1}{3}\arctan\frac{1}{3}x+C$ 22.) $\frac{1}{2}\arcsin^2 x+C$

Sample Problems - Solutions

Compute each of the following integrals. Please note that $\arcsin x$ is the same as $\sin^{-1} x$ and $\arctan x$ is the same as $\tan^{-1} x$.

1. $\displaystyle\int e^{-4x}\ dx$

   Solution: Let $u = -4x$. Then $du = -4dx$ and so $dx = -\dfrac{1}{4}du$. We now substitute in the integral

$$\int e^{-4x}\ dx = \int e^{u}\ -\frac{1}{4}du = -\frac{1}{4}\int e^{u}du = -\frac{1}{4}e^{u} + C = -\frac{1}{4}e^{-4x} + C$$

2. $\displaystyle\int_0^8 e^{-4x}\ dx$

   Solution:

$$\int_0^8 e^{-4x}\ dx = -\frac{1}{4}e^{-4x}\Big|_0^8 = -\frac{1}{4}\left(e^{-4(8)} - e^{-4(0)}\right) = -\frac{1}{4}\left(e^{-32} - e^{0}\right) = -\frac{1}{4}\left(e^{-32} - 1\right) = \frac{1}{4} - \frac{1}{4e^{32}}$$

3. $\displaystyle\int (x^2 - 2)(x^3 - 6x)^{207}\ dx$

   Solution: Let $u = x^3 - 6x$. Then $du = (3x^2 - 6)\,dx$ and so $dx = \dfrac{1}{(3x^2 - 6)}du$. We now substitute in the integral

$$\int \left(x^2 - 2\right)\left(x^3 - 6x\right)^{207}\ dx =$$

$$= \int \left(x^2 - 2\right)u^{207}\ \frac{1}{(3x^2 - 6)}du = \int \left(x^2 - 2\right)u^{207}\ \frac{1}{3\left(x^2 - 2\right)}du = \frac{1}{3}\int u^{207}\ du$$

$$= \frac{1}{3}\left(\frac{u^{208}}{208}\right) + C = \frac{1}{624}u^{208} + C = \frac{1}{624}\left(x^3 - 6x\right)^{208} + C$$

4. $\displaystyle\int e^{\cos x}\sin x\ dx$

   Solution: Let $u = \cos x$. Then $du = -\sin x dx$ and so $dx = -\dfrac{1}{\sin x}du$.

$$\int e^{\cos x}\sin x\ dx = \int e^{u}\sin x\ \frac{-1}{\sin x}du = -\int e^{u}du = -e^{u} + C = -e^{\cos x} + C$$

5. $\displaystyle\int_0^{\pi/2} \left(e^{\cos x}\sin x\right)\ dx$

Solution:

$$\int_0^{\pi/2} \left(e^{\cos x}\sin x\right)\ dx = -e^{\cos x}\Big|_0^{\pi/2} = -\left(e^{\cos(\pi/2)} - e^{\cos 0}\right) = -\left(e^0 - e^1\right) = e - 1$$

6. $\displaystyle\int \frac{3x}{\left(x^2+1\right)^7}\ dx$

Solution: Let $u = x^2 + 1$. Then $du = 2xdx$ and so $dx = \dfrac{1}{2x}du$.

$$\begin{aligned}\int \frac{3x}{\left(x^2+1\right)^7}\ dx &= \int \frac{3x}{u^7}\ \frac{1}{2x}du = \frac{3}{2}\int \frac{1}{u^7}\ du = \frac{3}{2}\int u^{-7}\ du = \frac{3}{2}\frac{u^{-6}}{-6} + C = -\frac{1}{4u^6} + C \\ &= -\frac{1}{4\left(x^2+1\right)^6} + C\end{aligned}$$

7. $\displaystyle\int \frac{12x^3}{3x^4+1}\ dx$

Solution: Let $u = 3x^4 + 1$. Then $du = 12x^3dx$ and so $dx = \dfrac{1}{12x^3}du$.

$$\int \frac{12x^3}{3x^4+1}\ dx = \int \frac{12x^3}{u}\ \frac{1}{12x^3}du = \int \frac{1}{u}\ du = \ln|u| + C = \ln\left(3x^4+1\right) + C$$

8. $\displaystyle\int (-3x+4)\, e^{-3x^2+8x}\ dx$

Solution: Let $u = -3x^2 + 8x$. Then $du = (-6x+8)\,dx$ and so $dx = \dfrac{1}{(-6x+8)}du$.

$$\begin{aligned}\int (-3x+4)\, e^{-3x^2+8x}\ dx &= \int (-3x+4)\, e^u\ \frac{1}{(-6x+8)}du = \int (-3x+4)\, e^u\ \frac{1}{2\,(-3x+4)}du \\ &= \frac{1}{2}\int e^u\ du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-3x^2+8x} + C\end{aligned}$$

9. $\displaystyle\int_0^2 (-3x+4)\, e^{-3x^2+8x}\ dx$

Solution:

$$\int_0^2 (-3x+4)\, e^{-3x^2+8x}\ dx = \frac{1}{2}e^{-3x^2+8x}\Big|_0^2 = \frac{1}{2}\left(e^{-3(2)^2+8(2)} - e^{-3(0)^2+8(0)}\right) = \frac{1}{2}\left(e^4 - e^0\right) = \frac{1}{2}\left(e^4 - 1\right)$$

10. $\displaystyle\int \frac{x+5}{x^2+1}\,dx$

Solution:

$$\int \frac{x+5}{x^2+1}\,dx = \int \left(\frac{x}{x^2+1} + \frac{5}{x^2+1}\right) dx = \int \frac{x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx$$

These two integrals can be computed via very different methods. For the first integral, let $u = x^2+1$. Then $du = 2x dx$ and so $dx = \dfrac{1}{2x} du$.

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\,\frac{1}{2x} du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(x^2+1\right) + C$$

and the second integral is

$$\int \frac{5}{x^2+1}\,dx = 5\int \frac{1}{x^2+1}\,dx = 5\arctan x + C$$

and so

$$\begin{aligned}\int \frac{x+5}{x^2+1}\,dx &= \int \frac{x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right) + C_1 + 5\arctan x + C_2 \\ &= \frac{1}{2}\ln\left(x^2+1\right) + 5\arctan x + C\end{aligned}$$

11. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

Solution: Let $u = e^x + 1$. Then $du = e^x dx$ and so $dx = \dfrac{1}{e^x} du$.

$$\int \frac{e^x}{e^x+1}\,dx = \int \frac{e^x}{u}\,\frac{1}{e^x} du = \int \frac{1}{u} du = \ln|u| + C = \ln\left(e^x+1\right) + C$$

12. $\displaystyle\int \cos x \sin^5 x\,dx$

Solution: Let $u = \sin x$. Then $du = \cos x dx$ and so $dx = \dfrac{1}{\cos x} du$.

$$\int \cos x \sin^5 x\,dx = \int \cos x\, u^5\,\frac{1}{\cos x} du = \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C$$

13. $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Solution: Let $u = x+1$. Then $du = dx$ and $x = u-1$

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{u-1}{\sqrt{u}}\,du = \int (u-1)\,u^{-1/2}\,du = \int uu^{-1/2} - u^{-1/2}\,du = \int u^{1/2} - u^{-1/2}\,du \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C\end{aligned}$$

14. $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\, dx$

Solution:

$$\begin{aligned}
\int_0^{15} \frac{x}{\sqrt{x+1}}\, dx &= \left(\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}\right)\Bigg|_0^{15} \\
&= \left(\frac{2}{3}(15+1)^{3/2} - 2(15+1)^{1/2}\right) - \left(\frac{2}{3}(0+1)^{3/2} - 2(0+1)^{1/2}\right) \\
&= \left(\frac{2}{3}\left(16^{3/2}\right) - 2\left(16^{1/2}\right)\right) - \left(\frac{2}{3}\left(1^{3/2}\right) - 2\left(1^{1/2}\right)\right) \\
&= \left(\frac{2}{3}(64) - 2(4)\right) - \left(\frac{2}{3} - 2\right) = \frac{128}{3} - 8 - \frac{2}{3} + 2 = \frac{126}{3} - 6 = 36
\end{aligned}$$

15. $\displaystyle\int_0^{3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

Solution: Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x})\, dx$ and so $dx = \dfrac{1}{(e^x - e^{-x})} du$. Also, when $x = 0$, then $u = e^0 + e^{-0} = 2$ and when $x = 3$, then $u = e^3 + e^{-3}$

$$\int_0^{3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int_2^{e^3+e^{-3}} \frac{e^x - e^{-x}}{u} \frac{1}{(e^x - e^{-x})} du = \int_2^{e^3+e^{-3}} \frac{1}{u}\, du = \ln|u|\Bigg|_2^{e^3+e^{-3}} = \ln\left(e^3 + e^{-3}\right) - \ln 2$$

16. $\displaystyle\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\, dx$

Solution: Let $u = \sqrt{x} + 1$. Then $du = \dfrac{1}{2\sqrt{x}} dx$ and so $dx = 2\sqrt{x} du$.

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\, dx = \int \frac{1}{\sqrt{x}\, u}\, 2\sqrt{x} du = 2\int \frac{1}{u}\, du = 2\ln|u| + C = 2\ln\left(\sqrt{x}+1\right) + C$$

17. $\displaystyle\int_0^{1} \frac{x+1}{x^2+1}\, dx$

Solution: We will first work out the indefinite integral. (For more details, pease refer to problem #10.)

$$\int \frac{x+1}{x^2+1}\, dx = \int \frac{x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx = \frac{1}{2}\ln\left(x^2+1\right) + \arctan x + C$$

and so the definite integral is.

$$\int_0^1 \frac{x+1}{x^2+1}\,dx = \left.\left(\frac{1}{2}\ln\left(x^2+1\right)+\arctan x\right)\right|_0^1$$
$$= \left(\frac{1}{2}\ln\left(1^2+1\right)+\arctan 1\right)-\left(\frac{1}{2}\ln\left(0^2+1\right)+\arctan 0\right)$$
$$= \left(\frac{1}{2}\ln 2+\frac{\pi}{4}\right)-0=\frac{1}{2}\ln 2+\frac{\pi}{4}$$

18. $\displaystyle\int_0^\infty xe^{-3x^2}\,dx$

Solution: We will first work out the indefinite integral. Let $u=-3x^2$. Then $du=-6xdx$ and so $dx=-\dfrac{1}{6x}du$.

$$\int xe^{-3x^2}\,dx=\int xe^u\;-\frac{1}{6x}du=-\frac{1}{6}\int e^u\,du=-\frac{1}{6}e^u+C=-\frac{1}{6}e^{-3x^2}+C$$

To compute the improper integral, we take the limit of definite integrals.

$$\int_0^\infty xe^{-3x^2}\,dx = \lim_{N\to\infty}\int_0^N xe^{-3x^2}\,dx$$
$$= \lim_{N\to\infty}\left(\left.-\frac{1}{6}e^{-3x^2}\right|_0^N\right)=\lim_{N\to\infty}\left(-\frac{1}{6}\left(e^{-3N^2}-e^{-3\left(0^2\right)}\right)\right)$$
$$= -\frac{1}{6}\lim_{N\to\infty}\left(e^{-3N^2}-1\right)=\frac{1}{6}\lim_{N\to\infty}\left(1-\frac{1}{e^{3N^2}}\right)=\frac{1}{6}$$

since $\displaystyle\lim_{N\to\infty}\frac{1}{e^{3N^2}}=0$.

19. $\displaystyle\int_0^\infty 3e^{-3x}\,dx$

Solution: We will first work out the indefinite integral. Let $u=-3x$. Then $du=-3dx$ and so $dx=-\dfrac{1}{3}du$.

$$\int 3e^{-3x}\,dx=\int 3e^u\;-\frac{1}{3}du=-\int e^u\,du=-e^u+C=-e^{-3x}+C$$

To compute the improper integral, we take the limit of definite integrals.

$$\int_0^\infty 3e^{-3x}\,dx=\lim_{N\to\infty}\int_0^N 3e^{-3x}\,dx=\lim_{N\to\infty}\left(\left.-e^{-3x}\right|_0^N\right)=\lim_{N\to\infty}\left(-\left(e^{-3N}-e^{-3(0)}\right)\right)=\lim_{N\to\infty}\left(1-\frac{1}{e^{3N}}\right)=1$$

since $\displaystyle\lim_{N\to\infty}\frac{1}{e^{3N}}=0$.

20. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Solution: Let $u=\sqrt{x}$. Then $du=\dfrac{1}{2\sqrt{x}}dx$ and so $dx=2\sqrt{x}du$.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx=\int \frac{\sin u}{\sqrt{x}}\,2\sqrt{x}du=2\int \sin u\,du=-2\cos u+C=-2\cos\sqrt{x}+C$$

21. $\displaystyle\int \frac{1}{x^2+9}\,dx$

Solution: The idea here is that we will 'get rid' of the 9 by factoring it out. If we used a substitution that would turn x^2 into $9u^2$, then the integrand would be $\dfrac{1}{9u^2+9}=\dfrac{1}{9}\cdot\dfrac{1}{u^2+1}$. We will pursue this substitution: $x^2=9u^2 \implies x=\pm 3u$. We select one of these.

Let $u=\dfrac{1}{3}x$. Then $du=\dfrac{1}{3}dx$ and so $dx=3du$.

$$\int \frac{1}{x^2+9}\,dx=\int \frac{1}{9u^2+9}\,3du=\frac{3}{9}\int \frac{1}{u^2+9}\,du=\frac{1}{3}\arctan u+C=\frac{1}{3}\arctan\frac{x}{3}+C$$

22. $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

Solution: Let $u=\arcsin x$. Then $du=\dfrac{1}{\sqrt{1-x^2}}dx$ and so $dx=\sqrt{1-x^2}du$.

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx=\int \frac{u}{\sqrt{1-x^2}}\,\sqrt{1-x^2}du=\int u\,du=\frac{u^2}{2}+C=\frac{1}{2}\left(\arcsin x\right)^2+C$$

