

1. The sum of two numbers is 31, their difference is 41. Find these numbers.

Solution: Let us call the smaller number x . Then the larger number is $x + 41$, since the difference between the two numbers is 41. The equation then is

$$\begin{array}{rcll}
 \underset{\text{smaller number}}{x} & + & \underset{\text{larger number}}{x + 41} & = 31 & \text{solve for } x \\
 & & 2x + 41 & = 31 & \text{subtract 41} \\
 & & 2x & = -10 & \text{divide by 2} \\
 & & x & = -5 &
 \end{array}$$

Thus the smaller number, labeled x is -5 . The larger number was labeled $x + 41$, so it must be $-5 + 41 = 36$. Thus the numbers are **-5 and 36** . We check: the difference between 36 and -5 is $36 - (-5) = 41$, and their sum is indeed $36 + (-5) = 31$. Thus our solution is correct.

2. The product of two numbers is 126. Their difference is 5. Find these numbers.

Solution: Let us label the smaller number as x . Then the larger number is $x + 5$. The equation is

$$\begin{array}{rcll}
 x(x + 5) & = & 126 & \text{Solve for } x \\
 x^2 + 5x & = & 126 & \\
 x^2 + 5x - 126 & = & 0 & \\
 (x + 14)(x - 9) & = & 0 & \\
 x_1 & = & -14 & \text{and } x_2 = 9
 \end{array}$$

If $x = -14$, then the larger number is $-14 + 5 = -9$. If $x = 9$, then the larger number is $9 + 5 = 14$. **The two solutions of the equation do not determine a pair of numbers: they are the smaller numbers in two pairs!** The answer is: **9 with 14 and -14 with -9** . We check in both cases: with 9 and 14

$$9(14) = 126 \checkmark \quad \text{and} \quad 14 - 9 = 5 \checkmark$$

With -14 and -9

$$-9(-14) = 126 \checkmark \quad \text{and} \quad -9 - (-14) = 5 \checkmark$$

3. One side of a rectangle is 3 ft shorter than twice the other side. Find the sides if the perimeter is 24 ft.

Solution: Let us denote the shorter side by x . Then the longer side is $2x - 3$. The equation expresses the perimeter.

$$\begin{array}{rcll}
 24 & = & 2(x) + 2(2x - 3) & \text{Solve for } x \\
 24 & = & 2x + 4x - 6 & \text{combine like terms} \\
 24 & = & 6x - 6 & \text{add 6} \\
 30 & = & 6x & \text{divide by 6} \\
 5 & = & x &
 \end{array}$$

Thus the shorter side is 5 ft, and the longer side is $2(5) - 3 = 7$ ft. Thus the answer is: **5 ft by 7 ft**. We check: 7 is indeed 3 less than twice 5, i.e. $7 = 2(5) - 3$ and the perimeter is $2(5) + 2(7) = 24$ ft. Thus our solution is indeed correct.

4. One side of a rectangle is 3 ft shorter than twice the other side. Find the sides if the area is 209 ft^2 .

Solution: Let us denote the shorter side by x . Then the longer side is $2x - 3$. The equation expresses the area.

$$\begin{aligned} x(2x - 3) &= 209 && \text{solve for } x \\ 2x^2 - 3x &= 209 \\ 2x^2 - 3x - 209 &= 0 \end{aligned}$$

We play the pq - game: we need two numbers, p and q such that

$$\begin{aligned} pq &= -418 \\ p + q &= -3 \end{aligned}$$

If the product pq has to be large but the sum is relatively small, like in this case, we should start looking close to $\sqrt{418}$, which is approximately $\sqrt{418} = 20.445$. We find that 19 and -22 work. We factor by grouping.

$$\begin{aligned} 2x^2 - 3x - 209 &= 0 \\ \underbrace{2x^2 + 19x}_{x(2x+19)} - \underbrace{22x - 209}_{-11(2x+19)} &= 0 \\ x(2x + 19) - 11(2x + 19) &= 0 \\ (x - 11)(2x + 19) &= 0 \\ x_1 &= 11 \quad \text{and} \quad x_2 = -\frac{19}{2} \end{aligned}$$

Since x denotes the side of a rectangle, which is a distance, and **distances are never negative**, the second solution, $x_2 = -\frac{19}{2}$ is immediately ruled out. If $x = 11$, the other side must be $2(11) - 3 = 19$. Thus the sides of the rectangle are **11 ft and 19 ft**. We check:

$$\begin{aligned} 2(11) - 3 &= 19 \checkmark \quad \text{and} \\ 11(19) &= 209 \checkmark \end{aligned}$$

Thus our solution is indeed correct.

5. One side of a rectangle is 4 in shorter than three times the other side. Find the sides if the perimeter of the rectangle is 48 in.

Solution: Let us denote the shorter side by x . Then the other side is $3x - 4$. The equation expresses the perimeter of the rectangle.

$$\begin{aligned} 2(x) + 2(3x - 4) &= 48 && \text{multiply out parentheses} \\ 2x + 6x - 8 &= 48 && \text{combine like terms} \\ 8x - 8 &= 48 && \text{add} \\ 8x &= 56 && \text{divide by 8} \\ x &= 7 \end{aligned}$$

If the shorter side was denoted by x , we now know it is 7 in. The longer side was denoted by $3x - 4$, so it must be $3(7) - 4 = 17$. Thus the sides of the rectangle are **7 in and 17 in**. We check: $P = 2(7 \text{ in}) + 2(17 \text{ in}) = 48 \text{ in}$ and $17 \text{ in} = 3(7 \text{ in}) - 4 \text{ in}$. Thus our solution is correct.

6. One side of a rectangle is 4 in shorter than three times the other side. Find the sides if the area of the rectangle is 319 in^2 .

Solution: Let us denote the shorter side by x . Then the other side is $3x - 4$. The equation expresses the area of the rectangle.

$$\begin{aligned} x(3x - 4) &= 319 && \text{multiply out parentheses} \\ 3x^2 - 4x &= 319 && \text{subtract 319} \\ 3x^2 - 4x - 319 &= 0 \end{aligned}$$

Because the equation is quadratic, we need to factor the left-hand side and then apply the zero property. We will factor by grouping. First we conduct the " pq -game". The sum of p and q has to be the linear coefficient (the number in front of x , with its sign), so it is -4 . The product of p and q has to be the product of the other coefficients, $3(-319) = -957$.

$$\begin{aligned} pq &= -957 \\ p + q &= -4 \end{aligned}$$

Now we need to find p and q . Because the product is negative, we're looking for a positive and a negative number. Because the sum is negative, the larger number must carry the negative sign. We enter $\sqrt{957}$ into the calculator and get a decimal:

$$\sqrt{957} = 30.935\dots$$

So we start looking for factors of 957, starting at 30, and moving down. We soon find 29 and -33 . These are our values for p and q . We use these numbers to express the linear term:

$$-4x = 29x - 33x$$

and factor by grouping.

$$\begin{aligned} 3x^2 - 4x - 319 &= 0 \\ \underbrace{3x^2 + 29x} + \underbrace{-33x - 319} &= 0 \\ x(3x + 29) - 11(3x + 29) &= 0 \\ (x - 11)(3x + 29) &= 0 \end{aligned}$$

We now apply the zero property. Either $x - 11 = 0$ or $3x + 29 = 0$. We solve both these equations for x .

$$\begin{aligned} x - 11 &= 0 \\ x &= 11 \end{aligned}$$

and

$$\begin{aligned} 3x + 29 &= 0 \\ 3x &= -29 \\ x &= -\frac{29}{3} \end{aligned}$$

Since distances can not be negative, the second solution for x , $-\frac{29}{3}$ is ruled out. Thus $x = 11$. Then the longer side is $3(11) - 4 = 29$, and so the rectangle's sides are **11 in** and **29 in** long. We check: $11 \text{ in}(29 \text{ in}) = 319 \text{ in}^2$ and $29 \text{ in} = 3(11 \text{ in}) - 4 \text{ in}$. Thus our solution is correct.