

Puzzle 3 - SOLUTIONS

Problem 1 *A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10, . . .). Then he clicks on every third number. (i.e. 3, 6, 9, 12, . . .) Now he is opening some doors, locking others. Then he clicks on every fourth number. (i.e. 4, 8, 12, 16,) Then on every fifth.... And so on, every sixth, every seventh, etc. Until every 100th; finally, he only clicks on the number 100. Then he orders that the prisoners that find their door open may go free. Who gets to go and who has to stay?*

Solution 1 *Instead of mentally repeating all steps, focus on a single cell. Say we are in cell 48. What happens to our door?*

We will get a click whenever the king clicks

on every 1st cell

on every 2nd cell

on every 3rd cell

on every 4th cell

on every 6th cell

on every 8th cell

on every 12th cell

on every 16th cell

on every 24th cell

on every 48th cell

10 clicks. It appears, we're staying. We make the following observations.

- 1. Every click corresponds to a factor of the cell's number. Thus the number of clicks on a cell equals the number of factors the cell's number.*
- 2. An even number of clicks means staying in prison. An odd number of factors means we're free.*

So the question can be rephrased: What numbers under 100 have an odd number of factors?

The question can be settled by just checking all numbers. We find that these numbers are

1, 4, 9, 16, 25, 36, 49, 64, 81, and 100

In other words, the square numbers.

It is true: Every number has an even number of divisors, except for the square numbers that have an odd number of divisors. The reason for that is that divisors always come in pairs. For example, 4 is a divisor of 48 because also 12 is: because $4 \cdot 12 = 48$. The way of obtaining an odd list is when one number is a pair with itself. Two examples are 98 and 81

	98			81	
1		98	1		81
2		49	3		27
7		14		9	