

1. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know how old your children are." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows how old the kids are. Do you?

Solution: The first information we have is that the product of the three kids' ages is 36. This gives us all possibilities. We need to find all possible ways we can express 36 as a product of 3 positive integers. Let us fix the age of the youngest kid.

If the youngest kid's age is 1, we have to express  $\frac{36}{1} = 36$  as a product of two integers.

| 36 |  |    |
|----|--|----|
| 1  |  | 36 |
| 2  |  | 18 |
| 3  |  | 12 |
| 4  |  | 9  |
| 6  |  | 6  |

Thus we have so far:

| youngest | middle | oldest |
|----------|--------|--------|
| 1        | 1      | 36     |
| 1        | 2      | 18     |
| 1        | 3      | 12     |
| 1        | 4      | 9      |
| 1        | 6      | 6      |

If the youngest kid's age is 2, we have to express  $\frac{36}{2} = 18$  as a product of two integers.

| 18 |  |    |
|----|--|----|
| 1  |  | 18 |
| 2  |  | 9  |
| 3  |  | 6  |

Notice that the first row is repetition of a previously found case. The youngest kid being 2 years old mean the middle one cannot be 1 year old.

Thus we have:

| youngest | middle | oldest |
|----------|--------|--------|
| 2        | 2      | 9      |
| 2        | 3      | 6      |

If the youngest kid's age is 3, we have to express  $\frac{36}{3} = 12$  as a product of two integers, where the smaller one is at least 3.

| 12 |  |   |
|----|--|---|
| 3  |  | 4 |

Thus we have:

| youngest | middle | oldest |
|----------|--------|--------|
| 3        | 3      | 4      |

Higher numbers will give us only repetition of previous cases. Thus we have all possibilities together:

| youngest | middle | oldest |
|----------|--------|--------|
| 1        | 1      | 36     |
| 1        | 2      | 18     |
| 1        | 3      | 12     |
| 1        | 4      | 9      |
| 1        | 6      | 6      |
| 2        | 2      | 9      |
| 2        | 3      | 6      |
| 3        | 3      | 4      |

B tells A the sum of the kid's ages. We can add all these tripples.

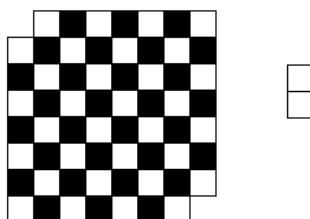
| youngest | middle | oldest | sum of ages |
|----------|--------|--------|-------------|
| 1        | 1      | 36     | 38          |
| 1        | 2      | 18     | 21          |
| 1        | 3      | 12     | 16          |
| 1        | 4      | 9      | 14          |
| 1        | 6      | 6      | 13          |
| 2        | 2      | 9      | 13          |
| 2        | 3      | 6      | 11          |
| 3        | 3      | 4      | 10          |

After B told A the sum of the ages, and A still didn't know the ages. This means that the sum was NOT ENOUGH information to distinguish two possibilities. Since the only sum that appears twice is 13, with 1, 6, 6 and 2, 2, 9, these are the only possibilities. (Any other sum was mentioned, A would have known the answer.) Since B said the YOUNGEST one has red hair, there is a youngest kid. This rules out 2, 2, 9 and so the kids are 1 and 6 and 6 years old.

2. We are at a cross road. One road leads into a dangerous swamp, the other road leads into a town. There is a pair of identical twins on the crossing. We know that one twin always tells the truth, the other twin always lies. We are allowed to ask only one question from only one brother. Is there a way to find out which road leads to town?

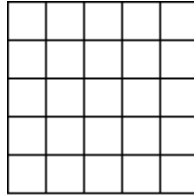
Solution: "If I asked your brother to point at the road to town, which road would he point at?" No matter, which brother we asked, either one would point out the road to danger.

3. Consider a chess board with two corners missing, as indicated on the picture below. We also have 31 pieces of domino, each of them can cover exactly 2 fields on the chess board. Is it possible to cover the chessboard with the domino pieces?



Solution: The solution boils down to coloring. No matter how we place a domino, it will always cover exactly one white and one black field. The original chess board had 32 white and 32 black fields. Since the two corners missing are both black, now we need to cover 32 white and 30 black fields with 31 dominos. This is obviously impossible.

4. There is a 5x5 board as the picture below shows. So happens, on each one of the fields there is a ladybug sitting. Suddenly, each decides to move to a neighboring field. (Two fields are neighbors if they have an edge in common.) Is it possible that after each have moved there is again exactly one lady bug sitting on each field?



Solution: This problem also boils down to coloring. Let us color the fields as if it was a chess board. Then we have 12 black and 13 white fields (or the other way around, 13 black and 12 white fields). Every bug that sat on a black field will sit on a white field, and every bug that sat on a white field will sit on a black field. Thus, it is impossible to have one bug on every field after the move.

5. Mr and Mrs Brown are having a party. They invited three other married couples, so there are eight people present. When greeting each other, some people shake hands with some people. Of course, nobody shakes hands with his/her spouse. When Mr Brown asks the other seven people: "How many people did you shake hands with?", he receives seven different answers. How many people did Mrs. Brown shake hands with?

Solution: Mr. Brown receives 7 different answers. First we need to figure out what those answers were. What is the highest possible number among the answers? Suppose I am the one who shook hands with everyone I could. There are 8 people, I won't shake my own hand, and I won't shake hands with my spouse, so that leaves 6 hand shakes for me.

What is the lowest possible number among the answers? Clearly 0. Now we know that the answers were 7 different numbers between 0 and 6. Those numbers can only be 0, 1, 2, 3, 4, 5, and 6. Let us represent the 4 married couples on the picture below.



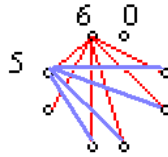
By symmetry, we may arbitrarily pick the person with 6 handshakes. We denote the number of handshakes by a little number. Let us represent the handshakes with lines. Now our picture looks like this:



Now we can see that our choice of the person with 6 handshakes determines the identity of the one with 0 handshakes: they have to be married! So now our picture is:



This is the main idea; we only need to apply it again and again. Let us pick the person with 5 hand shakes. Now the picture is:



Now we see that the person with 5 hand shakes and the person with 1 hand shake has to be married. And so on, we play this game until we find that the married couples are:

6 and 0,

5 and 1,

4 and 2.

The only number that was not paired up, is 3. This means that the person who said s/he shook hands with three people, is someone whose spouse was not asked. Mr. Brown did not ask himself, so his number is missing. So Mrs. Brown MUST BE the one with 3 hand shakes. Thus, Mrs. Brown shook hands with 3 people.

6. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10, . . . ). Then he clicks on every third number.(i.e. 3, 6, 9, 12, . . . ) Now he is opening some doors, locking others. Then he clicks on every fourth number. (i.e. 4, 8, 12, 16, . . . .) Then on every fifth.... And so on, every sixth, every seventh, etc. Until every 100th: which is, he only clicks on the number 100. Then he orders that the prisoners that find their door open may go free. Who gets to go and who has to stay?

Solution: Instead of mentally repeating all steps, focus on a single cell. Say we are in cell 48. What happens to our door?

Our door will get a click whenever the king clicks

|                   |                   |                    |                    |
|-------------------|-------------------|--------------------|--------------------|
| on every 1st cell | on every 4th cell | on every 12th cell | on every 48th cell |
| on every 2nd cell | on every 6th cell | on every 16th cell |                    |
| on every 3rd cell | on every 8th cell | on every 24th cell |                    |

10 clicks. It appears, we're staying. We make the following observations.

1.) Every click corresponds to a factor of the cell's number. Thus the number of clicks on a cell equals the number of factors the cell's number.

2.) An even number of clicks means staying in prison. An odd number of factors means we're free.

So the question can be rephrased: What numbers under 100 have an odd number of factors?

The question can be settled by just checking all numbers. We find that these numbers are

1, 4, 9, 16, 25, 36, 49, 64, 81, and 100

In other words, the square numbers. It is true: Every number has an even number of divisors, except for the square numbers that have an odd number of divisors. The reason for that is that divisors always come in pairs. For example, 4 is a divisor of 48 because also 12 is: because  $4 \cdot 12 = 48$ . The way of obtaining an odd list is when one number is a pair with itself. Two examples are 98 and 81

|   |    |    |   |    |    |
|---|----|----|---|----|----|
|   | 98 |    |   | 81 |    |
| 1 |    | 98 | 1 |    | 81 |
| 2 |    | 49 | 3 |    | 27 |
| 7 |    | 14 |   | 9  |    |

#### 7. Mind-reading. Instructions:

1. Think of a 5-6 digit number, that has at least two different digits in it. My example is 803225.
2. Create a second number by rearranging the digits of the previous number. My example is 320258.
3. Subtract the smaller number from the larger number. My example is  $803225 - 320258 = 482967$
4. Cross out any non-zero digit of the difference. My example is ~~4829~~67.
5. Announce the number you obtain by omitting the crossed out digit. My example is 48297.

If you tell the mindreader the last number, s/he can tell what digit you've crossed out. How?

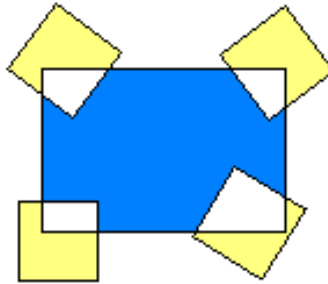
Solution: Not mind-reading. This trick is based on the rule of divisibility by 9. The rule is that every number's remainder after division by 9 is the same remainder if we divide the sum of its digits by 9. For example, the sum of the digits in the number 2500041 is  $2 + 5 + 4 + 1 = 12$ . When we divide 12 by 9 we get a remainder 3. This means that if we divided 2500041 by 9, we get a remainder 3. In particular, if the sum of the digits is divisible by 9, so is the number.

This trick exploits this property. When we create the second number, we use the same digits. So, we obtain two numbers that have the same sum of digits and thus the same remainder when divided by 9. This means that the third number, the difference of the first two, will certainly be divisible by 9 and so its digits add up to a number divisible by 9.

When someone announces their final number, we need to add the digits and see what number needs to be added to it to obtain something divisible by 9. (Now you see why crossing out 0 was not allowed, we wouldn't be able to distinguish it from 9.)

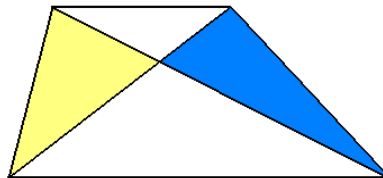
In my example, we hear 48297. Since  $4 + 8 + 2 + 9 + 7 = 30$ , we need add 6 to that to make it divisible by 9. (The next number, 15 is too large). Thus they crossed out a 6.

8. The picture below shows a rectangle (the sides' length are 2 and 3 unit long) and four identical squares (all four sides are 1 unit long). Determine which area is greater: the yellow or the blue?

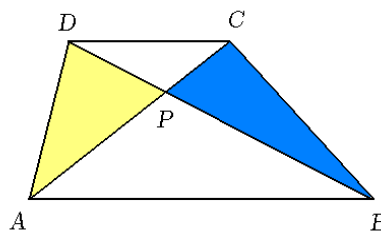


Solution: The area of the rectangle is  $6 \text{ unit}^2$ , and the area of the four squares is  $4 \text{ unit}^2$ . And, of course,  $6 \text{ unit}^2 > 4 \text{ unit}^2$ . While we don't know that blue or yellow area, they are  $6 - x$ , and  $4 - x$ , where  $x$  is the area of the white figures. Since we subtracted the same number from two quantities, the larger remains larger. So, the blue area is greater.

- b) The picture below shows a trapezoid. Which area is greater, the yellow or the blue?

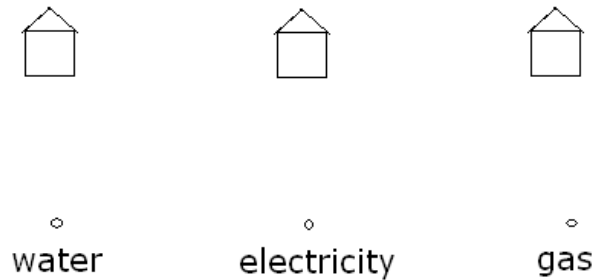


Solution: Let us denote the points as shown below.



Consider first the triangles  $ABD$  and  $ABC$ . These triangles have the same area because they have the same base and the same height. The yellow area was created by taking triangle  $ABD$  and removing triangle  $ABP$  from it. The blue area was created by taking triangle  $ABC$  and removing triangle  $ABP$  from it. So we started with equal areas and subtracted the same area, so they must remain the same. The blue and yellow areas are equal. The only thing we used is that  $ABCD$  is a trapezoid.

9. (This is a well known topology problem, and I heard it from my high school student, Eric.) Connect three buildings with the three utilities shown on the picture below. Each building has to be connected to all three utilities, in two dimensions, and no two utility pipes can cross each other.

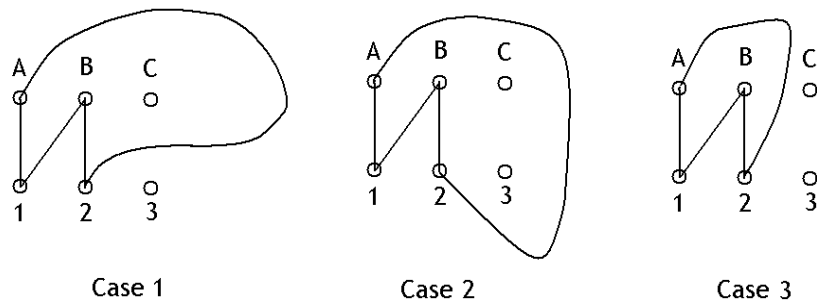


Solution: This is impossible. Here is one way to see it: Imagine that the six objects (3 houses and the 3 utility sources) are represented by 6 points on the plane. The main idea is that if we assume that all six are connected as described, we will always find 4 points (2 houses and 2 utilities) that are connected to each other and form a closed curve, while the 5th utility is inside that curve and the 6th one is outside. This means that the 5th and 6th can NOT be connected without crossing some wires.

Let us denote the houses by  $A$ ,  $B$  and  $C$  and the utilities by 1, 2, and 3. We will refer to the line connecting house  $A$  with utility 1 as  $A1$  or  $1A$ .

Consider now the following closed curve:  $1A - A2 - 2B - B1$ . This is clearly a closed curve. There are three possibilities:

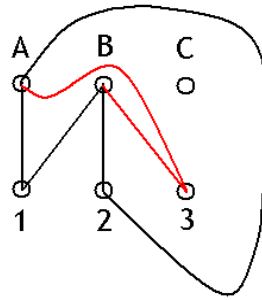
- 1.) The remaining two points,  $C$  and 3 are separated by this closed curve: one outside and one inside.
- 2.) The remaining two points,  $C$  and 3 are both inside this closed curve.
- 3.) The remaining two points,  $C$  and 3 are both outside this closed curve.



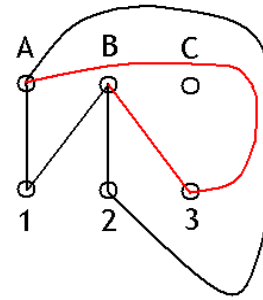
Case 1: We are done.  $C$  and 3 can not be connected without crossing the closed curve.

Case 2:  $C$  and 3 are both inside the closed curve  $1A - A2 - 2B - B1$ .

Consider now the closed curve  $1A - A3 - 3B - B1$ . Since 3 is inside the closed curve  $1A - A2 - 2B - B1$ , the entire curve  $1A - A3 - 3B - B1$  must be inside. Now there are two cases.



Case 2A

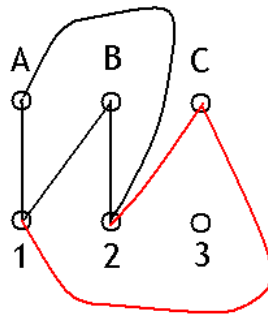


Case 2B

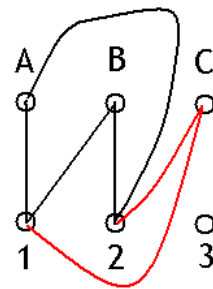
Case 2A.  $C$  is outside the closed curve  $1A - A3 - 3B - B1$ . Then we are done because  $C$  and 1 are separated by a closed curve, namely  $A2 - 2B - B3 - 3A$ .  $C$  is inside that curve and 1 is outside. Thus they cannot be connected without crossing a pipe.

Case 2B.  $C$  is inside the closed curve  $1A - A3 - 3B - B1$ . Then we are done because  $C$  and 2 are separated by a closed curve, namely  $A3 - 3B - B1 - 1A$ .  $C$  is inside that curve and 2 is outside. Thus they cannot be connected without crossing a pipe.

Case 3.  $C$  and 3 are both outside the closed curve  $1A - A2 - 2B - B1$ . Consider now the closed curve  $1B - B2 - 2C - C1$ . Because  $C$  is outside of the original curve  $1A - A2 - 2B - B1$ , the entire closed curve  $1B - B2 - 2C - C1$  must be outside. Now there are two cases: 3 might be inside the curve  $1B - B2 - 2C - C1$  or outside it.



Case 3A



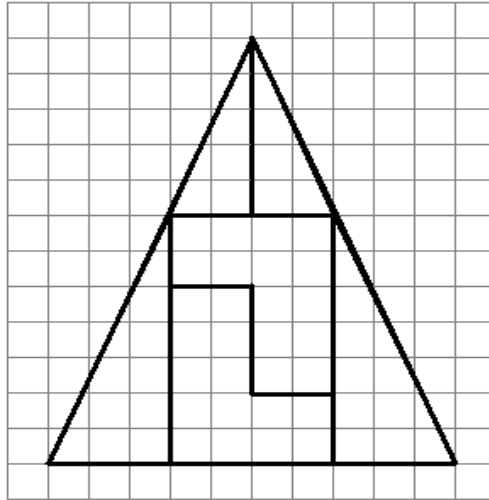
Case 3B

Case 3A: 3 is inside  $1B - B2 - 2C - C1$ . Then we are done because now this curve separates  $A$  and 3; 3 inside and  $A$  outside.

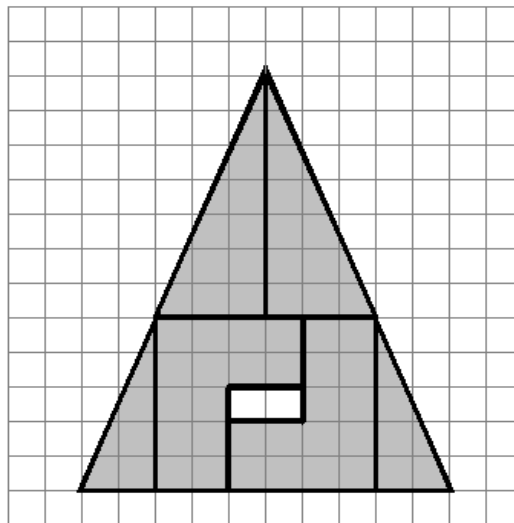
Case 3B: 3 is outside  $1B - B2 - 2C - C1$ . Then we are done because now the curve  $1A - A2 - 2C - C1$  separates  $B$  and 3;  $B$  inside and 3 outside.

This completes our proof.

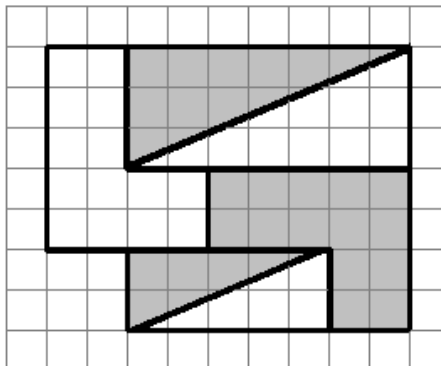
10. (Paul Curry) Imagine that we cut the figure below out of paper. The area of the triangle is  $A = \frac{10 \cdot 12}{2} = 60$  unit<sup>2</sup>.



Suppose we color the other side of the paper, turn the pieces upside down, and rearrange them to obtain the figure shown below.



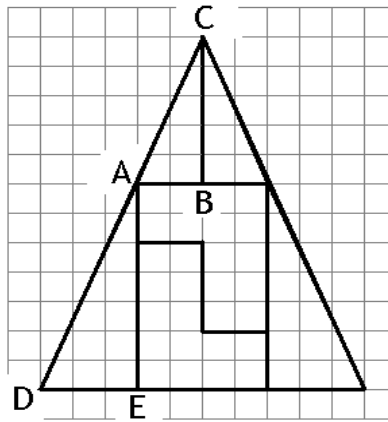
Now the area appears to be 58 unit<sup>2</sup> since there is a 2 unit<sup>2</sup> area of a whole that the triangle developed. To make matters worse, we now again rearrange the pieces, turning some of them back to the original side.



The area of this figure is  $7(9) - 4 = 59 \text{ unit}^2$ .

Logic tells us that areas of figures do not change if we rearrange them or turn them on their other side. So, what is wrong with these pictures, and how much is really the area of these shapes,  $58 \text{ unit}^2$ ,  $59 \text{ unit}^2$ , or  $60 \text{ unit}^2$ ?

Solution: Consider the original picture as shown below. Although the shape looks like a triangle, it is actually not! It is a pentagon because  $DA$  and  $AC$  are two different lines and not one. Consider first line  $AC$ . From the right triangle  $ABC$ , we can see that the slope of  $AC$  is  $\frac{5}{2}$ . Similarly, from triangle  $DEA$ , line  $DA$  has slope  $\frac{7}{3}$ . Since  $\frac{5}{2}$  and  $\frac{7}{3}$  are different numbers  $2\frac{1}{2}$  and  $2\frac{1}{3}$ , the two lines can not be parallel or identical.



If the lines we use to draw the picture are thick enough, they cover up this fact and make the pentagon appear as a triangle. The area of this shape can not be computed as one large triangle, so the first and the second computations are incorrect. Instead, we compute the area of each part separately and obtain the answer  $59 \text{ unit}^2$ .

11. A bus makes a roundtrip between towns A and B. From A to B, the bus completes the trip with an average speed of  $60 \frac{\text{mi}}{\text{hr}}$ . From B to A, it travels with an average velocity of  $40 \frac{\text{mi}}{\text{hr}}$ . What is the average speed of the bus for the entire roundtrip? (Hint: it is NOT the average of 40 and 60)

Solution: Suppose the towns are 120 miles apart. We can use any numbers, but 120 is 'nice' because it is divisible by 40 and 60. The time it took to travel from A to B was

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{120 \text{ mi}}{60 \frac{\text{mi}}{\text{hr}}} = 2\text{hr}$$

The trip from B to A:

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{120 \text{ mi}}{40 \frac{\text{mi}}{\text{hr}}} = 3\text{hr}$$

The average velocity for the round trip is

$$v = \frac{\text{total distance}}{\text{total time}} = \frac{240 \text{ mi}}{5 \text{ hr}} = 48 \frac{\text{mi}}{\text{hr}}$$

Thus the average velocity is  $48 \frac{\text{mi}}{\text{hr}}$ . Note: We can obtain these results without making up a distance. Denote it by  $s$ . Then the computation works like this. The time it took to travel from A to B was

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{s}{60}$$

The trip from B to A:

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{s}{40}$$

The average velocity for the round trip is

$$v = \frac{\text{total distance}}{\text{total time}} = \frac{s + s}{\frac{s}{60} + \frac{s}{40}} = \frac{2s}{\frac{2s}{120} + \frac{3s}{120}} = \frac{2s}{\left(\frac{5s}{120}\right)} = 2s \cdot \frac{120}{5s} = \frac{240s}{5s} = 48$$

Thus the average velocity is  $48 \frac{\text{mi}}{\text{hr}}$ .

12. There is a very difficult track in a mountain area, where the world record of driving a lap is 15 miles per hour average speed. A race car driver announces that he intends to beat this record and run the course with an average speed of 30 miles per hour. As he is driving, his progress is monitored. At the half of the track, his average speed was 15 miles per hour. In spite of this, he arrives at the finish just a few seconds too late to have an average speed of 30 miles per hour. His funeral was three days later. How did he die?

Solution: He fall from a very tall mountain. This problem is about time. Let us denote the length of the path by  $s$  (making up a number would also work). If the race car driver pledged to complete the path with an average speed of 30 miles per hour, that means that he will have to finish within the time frame of

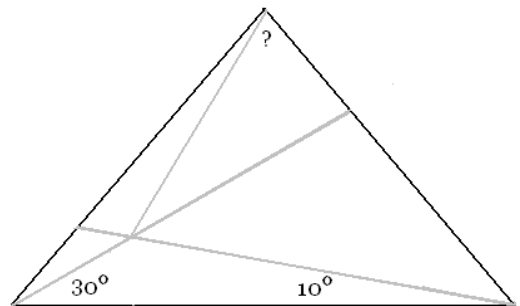
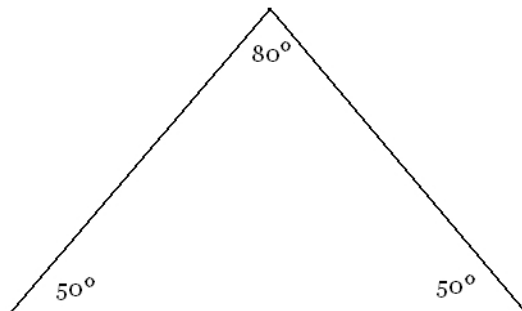
$$t = \frac{s}{v} = \frac{s}{30}$$

When halfway through the path his average speed is 15 miles per hour, that means that he used up all his time, because

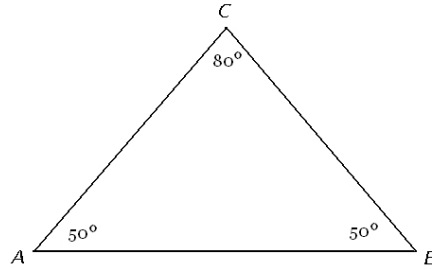
$$t = \frac{\left(\frac{s}{2}\right)}{15} = \frac{s}{30}$$

Yet, he arrived to the finish line just a few seconds too late to accomplish what he set out to do. This means that he completed the entire second half of the trip in just a few seconds. In a difficult, mountainous terrain this can only mean that he fell from high up in the mountain (probably at the top) to the finish line which was located at the feet of the mountain.

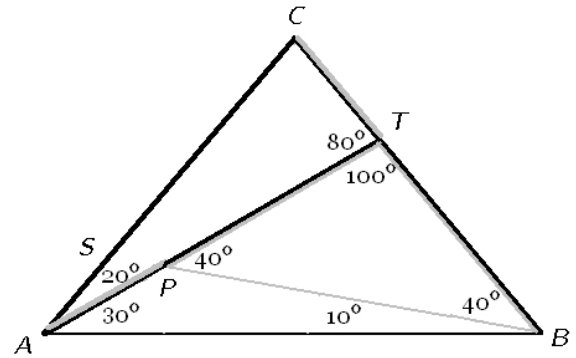
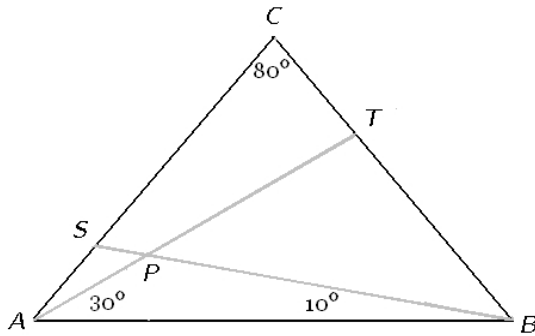
13. (From my student, Ramon Gozales.) The triangle below has angles  $50^\circ$ ,  $50^\circ$ , and  $80^\circ$  as shown on the picture below. We measure up  $10^\circ$  and  $30^\circ$  from the base as shown on the picture below. Find the measure of the angle shown on the picture.



Solution: Based on the angles, triangle  $ABC$  is isosceles and so  $\overline{AC} = \overline{BC}$ .



Let us now draw in the lines as described in the problem.

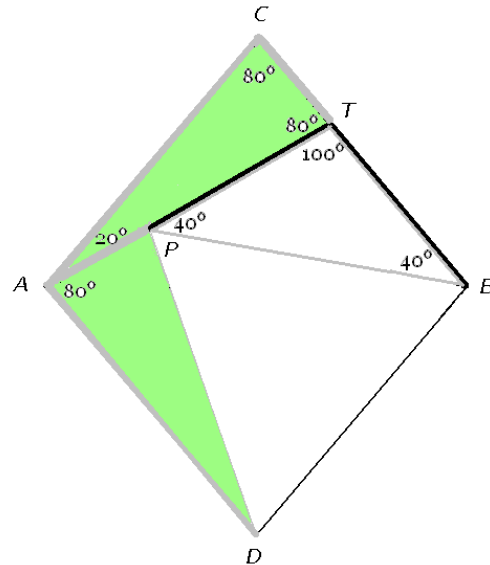
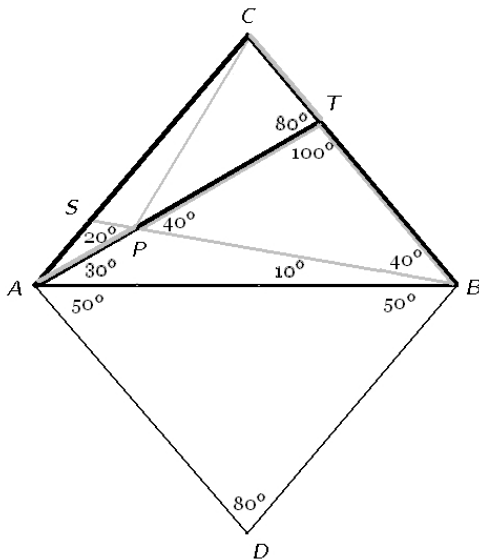


Because  $\angle CAB = 50^\circ$  and  $\angle TAC = 30^\circ$ , clearly  $\angle CAT = 20^\circ$ . Consider now triangle  $ATC$ .  $\angle CAT = 20^\circ$  and  $\angle TCA = 80^\circ$  forces the third angle,  $\angle ATC$  to be also  $80^\circ$ . Thus triangle  $ATC$  is isosceles and so  $\overline{AT} = \overline{AC}$ . Also  $\overline{AC} = \overline{BC} = \overline{AT}$ .

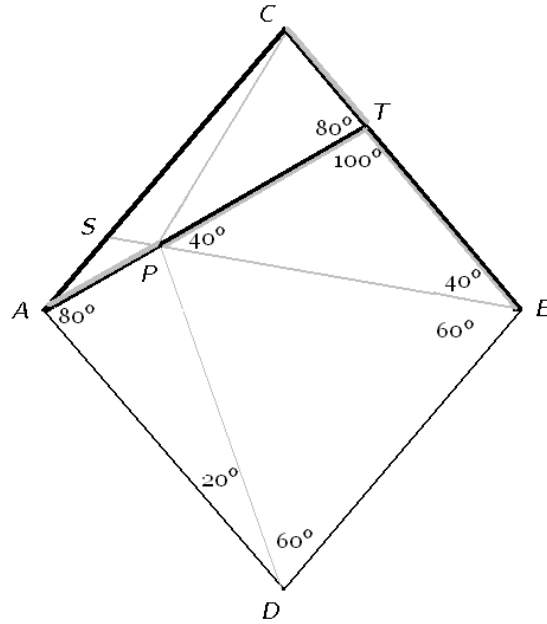
Consider now triangle  $BPT$ . Clearly  $\angle PTB = 100^\circ$  and  $\angle TBP = 40^\circ$  which forces the third angle,  $\angle TPB$  to be also  $40^\circ$ . Thus triangle  $BPT$  is isosceles and  $\overline{PT} = \overline{TB}$ .

Next we claim that  $\overline{AP} = \overline{TC}$ . This is because  $\overline{AT} = \overline{AC}$  and  $\overline{PT} = \overline{TB}$  and so  $\overline{AP} = \overline{AT} - \overline{PT} = \overline{AC} - \overline{TB} = \overline{TC}$ .

Let us now reflect the triangle to its side  $AB$ .



We now claim that triangles  $ACT$  and  $ADP$  are congruent. This is because  $\overline{TC} = \overline{AP}$ ,  $\overline{AT} = \overline{AD}$  and  $\angle CTA = \angle PAD = 80^\circ$ . Because of this, we know that line segment  $PD$  splits the angle at point  $D$  into  $20^\circ$  and  $60^\circ$ .



We now claim that triangle  $BPD$  is equilateral, because it has two angles measuring  $60^\circ$ . Consequently,  $\overline{BP} = \overline{PD} = \overline{DB}$ . These line segments are also of the same length as  $\overline{AC} = \overline{CB}$ .

This means that  $\overline{BP} = \overline{CB}$  and so triangle  $BPC$  is isosceles. Then the opposite angles are also equal and so  $\angle CPB = \angle BCP$ . Because the third angle is  $40^\circ$ , this forces these angles to measure  $70^\circ$ . Thus  $\angle BCP = 70^\circ$ .